Lipschitz realcompactifications and functionals on Lipschitz spaces

Ramón J. Aliaga

Universidad Politécnica de Valencia

raalva@upv.es

(based on joint work with E. Pernecká and R. J. Smith)

New perspectives in Banach spaces and Banach lattices
Castro Urdiales, 8 Jul 2024

Lipschitz spaces

Let (M, d) be a complete metric space.
Fix a **base point** $0 \in M$.

Given $f : M \to \mathbb{R}$, its [optimal] Lipschitz constant is

$$\|f\|_L = \sup \left\{ \frac{f(x) - f(y)}{d(x,y)} : x \neq y \in M \right\}$$

Lipschitz spaces

Let (M,d) be a complete metric space.
Fix a **base point** $0 \in M$.

Given $f : M \to \mathbb{R}$, its [optimal] Lipschitz constant is

$$\|f\|_L = \sup\left\{\frac{f(x)-f(y)}{d(x,y)} : x \neq y \in M\right\}$$

We consider the vector spaces

$$\begin{aligned}
\mathrm{Lip}(M) &= \{f : M \to \mathbb{R} \ : \ \|f\|_L < \infty\} \\
\mathrm{Lip}_b(M) &= \{f \in \mathrm{Lip}(M) \ : \ f \text{ bounded}\} \\
\mathrm{Lip}_0(M) &= \{f \in \mathrm{Lip}(M) \ : \ f(0) = 0\}
\end{aligned}$$

The **Lipschitz space** is the Banach space $(\mathrm{Lip}_0(M), \|\cdot\|_L)$.

Functionals on Lipschitz spaces

We consider the **dual Lipschitz space** $\mathrm{Lip}_0(M)^*$. It contains the evaluation functionals $\delta(x) : f \mapsto f(x)$, $x \in M$.

The mapping $\delta : M \to \mathrm{Lip}_0(M)^*$ is an (nonlinear) isometric embedding.

Functionals on Lipschitz spaces

We consider the **dual Lipschitz space** $\mathrm{Lip}_0(M)^*$. It contains the evaluation functionals $\delta(x) : f \mapsto f(x)$, $x \in M$.

The mapping $\delta : M \to \mathrm{Lip}_0(M)^*$ is an (nonlinear) isometric embedding.

The **Lipschitz-free space** over M is

$$\mathcal{F}(M) = \overline{\mathrm{span}}\,\delta(M) \subset \mathrm{Lip}_0(M)^*$$

Functionals on Lipschitz spaces

We consider the **dual Lipschitz space** $\mathrm{Lip}_0(M)^*$. It contains the evaluation functionals $\delta(x): f \mapsto f(x)$, $x \in M$.

The mapping $\delta : M \to \mathrm{Lip}_0(M)^*$ is an (nonlinear) isometric embedding.

The **Lipschitz-free space** over M is

$$\mathcal{F}(M) = \overline{\mathrm{span}}\, \delta(M) \subset \mathrm{Lip}_0(M)^*$$

Theorem *(Arens, Eells 1956)*

$\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$, and the weak* topology on $B_{\mathrm{Lip}_0(M)}$ is the topology of pointwise convergence.

Non-weak*-continuous functionals

If M is infinite then $\mathrm{Lip}_0(M)^* \neq \mathcal{F}(M)$.

Non-weak*-continuous functionals

If M is infinite then $\mathrm{Lip}_0(M)^* \neq \mathcal{F}(M)$.

Examples in $\mathrm{Lip}_0(M)^* \setminus \mathcal{F}(M)$:

1. If $x \in M$ is an accumulation point, there are **derivations at** x, e.g.

$$\text{weak}^* \text{ cluster points of } \quad \frac{\delta(x) - \delta(x_n)}{d(x, x_n)} \in S_{\mathcal{F}(M)} \quad \text{where } x_n \to x$$

Non-weak*-continuous functionals

If M is infinite then $\mathrm{Lip}_0(M)^* \neq \mathcal{F}(M)$.

Examples in $\mathrm{Lip}_0(M)^* \setminus \mathcal{F}(M)$:

1. If $x \in M$ is an accumulation point, there are **derivations at** x, e.g.

$$\text{weak}^* \text{ cluster points of } \quad \frac{\delta(x) - \delta(x_n)}{d(x, x_n)} \in S_{\mathcal{F}(M)} \quad \text{where } x_n \to x$$

2. Let βM be the Stone-Čech compactification of M.
If M is bounded, then for $\xi \in \beta M \setminus M$ the functional

$$\delta(\xi) : f \mapsto f(\xi)$$

is in $\mathrm{Lip}_0(M)^* \setminus \mathcal{F}(M)$.

Functionals that avoid infinity/the base point

We say that $\phi \in \mathrm{Lip}_0(M)^*$ **avoids infinity** if "it is determined by functions with bounded support". That is: for all $f \in \mathrm{Lip}_0(M)$

$$\phi(f_n) \to \phi(f) \qquad \text{where} \qquad f_n = \begin{cases} f \text{ on } B(0,n) \\ 0 \text{ outside } B(0,2n) \\ \text{smooth in between} \end{cases}$$

Functionals that avoid infinity/the base point

We say that $\phi \in \mathrm{Lip}_0(M)^*$ **avoids infinity** if "it is determined by functions with bounded support". That is: for all $f \in \mathrm{Lip}_0(M)$

$$\phi(f_n) \to \phi(f) \qquad \text{where} \qquad f_n = \begin{cases} f \text{ on } B(0,n) \\ 0 \text{ outside } B(0,2n) \\ \text{smooth in between} \end{cases}$$

We say that $\phi \in \mathrm{Lip}_0(M)^*$ **avoids the base point** if "it is determined by functions that vanish near the base point". That is: for all $f \in \mathrm{Lip}_0(M)$

$$\phi(f_n) \to \phi(f) \qquad \text{where} \qquad f_n = \begin{cases} f \text{ outside } B(0,\frac{1}{n}) \\ 0 \text{ on } B(0,\frac{1}{2n}) \\ \text{smooth in between} \end{cases}$$

Functionals that avoid infinity/the base point

We say that $\phi \in \mathrm{Lip}_0(M)^*$ **avoids infinity** if "it is determined by functions with bounded support". That is: for all $f \in \mathrm{Lip}_0(M)$

$$\phi(f_n) \to \phi(f) \qquad \text{where} \qquad f_n = \begin{cases} f \text{ on } B(0,n) \\ 0 \text{ outside } B(0,2n) \\ \text{smooth in between} \end{cases}$$

We say that $\phi \in \mathrm{Lip}_0(M)^*$ **avoids the base point** if "it is determined by functions that vanish near the base point". That is: for all $f \in \mathrm{Lip}_0(M)$

$$\phi(f_n) \to \phi(f) \qquad \text{where} \qquad f_n = \begin{cases} f \text{ outside } B(0,\frac{1}{n}) \\ 0 \text{ on } B(0,\frac{1}{2n}) \\ \text{smooth in between} \end{cases}$$

All elements of $\mathcal{F}(M)$ avoid infinity and the base point.

The uniform compactification

We build a compactification that is more useful than Stone-Čech βM when dealing with Lipschitz functions.

The uniform compactification

We build a compactification that is more useful than Stone-Čech βM when dealing with Lipschitz functions.

Consider the compact Hausdorff space

$$P = \prod_{f \in \mathrm{Lip}_b(M)} \overline{f(M)} \quad \subset \quad \mathbb{R}^{\mathrm{Lip}_b(M)}$$

Then $e\colon x \mapsto (f(x))_{f \in \mathrm{Lip}_b(M)}$ is a homeomorphic embedding of M into P.

The **uniform** (or **Samuel**, or **Smirnov**) **compactification** of M is

$$M^{\mathcal{U}} = \overline{e(M)}^{P} \quad \subset \quad P$$

where we identify M with $e(M)$.

For $A \subset M^{\mathcal{U}}$, we denote $\overline{A}^{\mathcal{U}}$ its closure in $M^{\mathcal{U}}$.

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.
2. In $M^{\mathcal{U}}$, $\xi_i \to \xi$ iff $f^{\mathcal{U}}(\xi_i) \to f^{\mathcal{U}}(\xi)$ for all $f \in \mathrm{Lip}_b(M)$.

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.
2. In $M^{\mathcal{U}}$, $\xi_i \to \xi$ iff $f^{\mathcal{U}}(\xi_i) \to f^{\mathcal{U}}(\xi)$ for all $f \in \mathrm{Lip}_b(M)$.
3. **Given $A, B \subset M$, we have $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$ iff $d(A,B) > 0$.**

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.
2. In $M^{\mathcal{U}}$, $\xi_i \to \xi$ iff $f^{\mathcal{U}}(\xi_i) \to f^{\mathcal{U}}(\xi)$ for all $f \in \mathrm{Lip}_b(M)$.
3. **Given $A, B \subset M$, we have $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$ iff $d(A,B) > 0$.**
4. Property (3) characterizes $M^{\mathcal{U}}$ among all compactifications of M.

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.
2. In $M^{\mathcal{U}}$, $\xi_i \to \xi$ iff $f^{\mathcal{U}}(\xi_i) \to f^{\mathcal{U}}(\xi)$ for all $f \in \mathrm{Lip}_b(M)$.
3. **Given $A, B \subset M$, we have $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$ iff $d(A,B) > 0$.**
4. Property (3) characterizes $M^{\mathcal{U}}$ among all compactifications of M.

Proof of (3):

- Suppose $d(A,B) > 0$.
  Then there is $f \in \mathrm{Lip}_b(M)$ such that $f\restriction_A = 0, f\restriction_B = 1$.
  Therefore $f^{\mathcal{U}}(\xi) = 0$ for $\xi \in \overline{A}^{\mathcal{U}}$ and $f^{\mathcal{U}}(\xi) = 1$ for $\xi \in \overline{B}^{\mathcal{U}}$.
  Thus $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$.

Properties

Properties of $M^{\mathcal{U}}$:

1. Every $f \in \mathrm{Lip}_b(M)$ can be extended to $f^{\mathcal{U}} \in C_b(M^{\mathcal{U}})$.
2. In $M^{\mathcal{U}}$, $\xi_i \to \xi$ iff $f^{\mathcal{U}}(\xi_i) \to f^{\mathcal{U}}(\xi)$ for all $f \in \mathrm{Lip}_b(M)$.
3. **Given $A, B \subset M$, we have $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$ iff $d(A,B) > 0$.**
4. Property (3) characterizes $M^{\mathcal{U}}$ among all compactifications of M.

Proof of (3):

- Suppose $d(A,B) > 0$.
  Then there is $f \in \mathrm{Lip}_b(M)$ such that $f\restriction_A = 0$, $f\restriction_B = 1$.
  Therefore $f^{\mathcal{U}}(\xi) = 0$ for $\xi \in \overline{A}^{\mathcal{U}}$ and $f^{\mathcal{U}}(\xi) = 1$ for $\xi \in \overline{B}^{\mathcal{U}}$.
  Thus $\overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}} = \varnothing$.
- Suppose $d(A,B) = 0$.
  Then there are $a_n \in A$, $b_n \in B$ with $d(a_n, b_n) \to 0$.
  Let α, β be cluster points of (a_n), (b_n) in $M^{\mathcal{U}}$.
  For every $f \in \mathrm{Lip}_b(M)$ we get $f(a_n) - f(b_n) \to 0$ and thus $f^{\mathcal{U}}(\alpha) = f^{\mathcal{U}}(\beta)$.
  It follows $\alpha = \beta \in \overline{A}^{\mathcal{U}} \cap \overline{B}^{\mathcal{U}}$.

The Lipschitz realcompactification

$\mathrm{Lip}_b(M)$ functions can be extended to all of $M^{\mathcal{U}}$.
Unbounded $\mathrm{Lip}(M)$ functions can only be extended to points "not at infinity", i.e. limits of bounded nets in M.

The Lipschitz realcompactification

$\mathrm{Lip}_b(M)$ functions can be extended to all of $M^{\mathcal{U}}$.
Unbounded $\mathrm{Lip}(M)$ functions can only be extended to points "not at infinity", i.e. limits of bounded nets in M.

The **Lipschitz realcompactification** of M is the set of such points

$$M^{\mathcal{R}} = \bigcup_{n=1}^{\infty} \overline{B(0,n)}^{\mathcal{U}} \quad \subset \quad M^{\mathcal{U}}$$

Recall: a topological space is **realcompact** if it is homeomorphic to a closed subset of a Cartesian product of real lines.

The Lipschitz realcompactification

$\mathrm{Lip}_b(M)$ functions can be extended to all of $M^{\mathcal{U}}$.
Unbounded $\mathrm{Lip}(M)$ functions can only be extended to points "not at infinity", i.e. limits of bounded nets in M.

The **Lipschitz realcompactification** of M is the set of such points

$$M^{\mathcal{R}} = \bigcup_{n=1}^{\infty} \overline{B(0,n)}^{\mathcal{U}} \quad \subset \quad M^{\mathcal{U}}$$

Recall: a topological space is **realcompact** if it is homeomorphic to a closed subset of a Cartesian product of real lines.

Proposition *(Aliaga, Pernecká, Smith 2024)*

$\delta : M^{\mathcal{R}} \to (\mathrm{Lip}_0(M)^*, w^*)$ is a homeomorphic embedding, and $\delta(M^{\mathcal{R}})$ is weak* closed in $\mathrm{Lip}_0(M)^*$.

A metric for the realcompactification

For $\xi, \zeta \in M^{\mathcal{R}}$ define

$$\bar{d}(\xi, \zeta) = \|\delta(\xi) - \delta(\zeta)\|_{\mathrm{Lip}_0(M)^*}$$

- $(M^{\mathcal{R}}, \bar{d})$ is a complete metric space
- M is a closed subset of $(M^{\mathcal{R}}, \bar{d})$
- $\bar{d}$ is lower semicontinuous
- Closed $\bar{d}$-balls are compact
- $\mathcal{F}(M^{\mathcal{R}}, \bar{d}) = \overline{\mathrm{span}}\, \delta(M^{\mathcal{R}}) \subset \mathrm{Lip}_0(M)^*$

Proposition

Let $A \subset M^{\mathcal{R}}$ be closed, and $\varphi \in \mathrm{Lip}_b(A, \bar{d})$ be continuous. Then $\varphi = f^{\mathcal{U}}\restriction_A$ for some $\varphi \in \mathrm{Lip}_b(M)$ with the same Lipschitz constant.

The metric bidual

$(M^{\mathcal{R}}, \bar{d})$ is the "metric bidual" of (M, d).
Does it satisfy a metric Principle of Local Reflexivity?

The metric bidual

$(M^{\mathcal{R}}, \bar{d})$ is the "metric bidual" of (M, d).
Does it satisfy a metric Principle of Local Reflexivity?

Question

Is M an almost isometric local retract of $(M^{\mathcal{R}}, \bar{d})$?
That is, for every finite $E \subset M^{\mathcal{R}}$ and $\varepsilon > 0$ there is a map $r : E \to M$ such that

$$(1-\varepsilon)\bar{d}(\xi,\zeta) \leq d(r(\xi), r(\zeta)) \leq (1+\varepsilon)\bar{d}(\xi,\zeta)$$

and $r(\xi) = \xi$ for $\xi \in M$.

The metric bidual

$(M^{\mathcal{R}}, \bar{d})$ is the "metric bidual" of (M, d).
Does it satisfy a metric Principle of Local Reflexivity?

Question

Is M an almost isometric local retract of $(M^{\mathcal{R}}, \bar{d})$?
That is, for every finite $E \subset M^{\mathcal{R}}$ and $\varepsilon > 0$ there is a map $r : E \to M$ such that

$$(1-\varepsilon)\bar{d}(\xi,\zeta) \leq d(r(\xi), r(\zeta)) \leq (1+\varepsilon)\bar{d}(\xi,\zeta)$$

and $r(\xi) = \xi$ for $\xi \in M$.

A consequence would be: Given $x \neq y \in M$, TFAE:

- x and y are **discretely connectable**, that is, for every $\varepsilon > 0$ there are points $p_1 = x, p_2, \ldots, p_{n-1}, p_n = y$ in M such that
  - $d(p_k, p_{k+1}) \leq \varepsilon$,
  - $d(p_1, p_2) + \ldots + d(p_{n-1}, p_n) \leq d(x, y) + \varepsilon$
- x and y are connected by a geodesic in $(M^{\mathcal{R}}, \bar{d})$.

Application 1: Defining supports

The **support** of $m \in \mathcal{F}(M)$ is

$$\operatorname{supp}(m) = \bigcap \{A \subset M \text{ closed} : m(f) = 0 \text{ if } f\restriction_A = 0\}$$

It is a closed separable subset of M where "m is concentrated".

Application 1: Defining supports

The **support** of $m \in \mathcal{F}(M)$ is

$$\operatorname{supp}(m) = \bigcap \{A \subset M \text{ closed} : m(f) = 0 \text{ if } f\restriction_A = 0\}$$

It is a closed separable subset of M where "m is concentrated".

Theorem *(Aliaga, Pernecká 2020)*

If $m \in \mathcal{F}(M)$ then

$$f\restriction_{\operatorname{supp}(m)} = 0 \quad \Longrightarrow \quad m(f) = 0$$

In particular, $\operatorname{supp}(m) = \varnothing$ iff $m = 0$.

Extended supports

We would like to define supports in $\mathrm{Lip}_0(M)^*$ in a meaningful way (e.g. the support of $\delta(\xi)$, $\xi \in M^{\mathcal{R}}$ should be $\{\xi\}$).

Extended supports

We would like to define supports in $\mathrm{Lip}_0(M)^*$ in a meaningful way (e.g. the support of $\delta(\xi)$, $\xi \in M^{\mathcal{R}}$ should be $\{\xi\}$).

The **extended support** of $\phi \in \mathrm{Lip}_0(M)^*$ is

$$\mathcal{S}(\phi) = \bigcap \{A \subset M^{\mathcal{U}} \text{ closed} : \phi(f) = 0 \text{ if } f^{\mathcal{U}}\restriction_A = 0\}$$

If $\phi \in \mathcal{F}(M)$ then $\mathcal{S}(\phi) = \overline{\mathrm{supp}(\phi)}^{\mathcal{U}}$ and $\mathrm{supp}(\phi) = \mathcal{S}(\phi) \cap M$.

Extended supports

We would like to define supports in $\mathrm{Lip}_0(M)^*$ in a meaningful way (e.g. the support of $\delta(\xi)$, $\xi \in M^{\mathcal{R}}$ should be $\{\xi\}$).

The **extended support** of $\phi \in \mathrm{Lip}_0(M)^*$ is

$$\mathcal{S}(\phi) = \bigcap \{A \subset M^{\mathcal{U}} \text{ closed} : \phi(f) = 0 \text{ if } f^{\mathcal{U}}\restriction_A = 0\}$$

If $\phi \in \mathcal{F}(M)$ then $\mathcal{S}(\phi) = \overline{\mathrm{supp}(\phi)}^{\mathcal{U}}$ and $\mathrm{supp}(\phi) = \mathcal{S}(\phi) \cap M$.

⚠ It is **not** true that $f^{\mathcal{U}}|_{\mathcal{S}(\phi)} = 0$ implies $\phi(f) = 0$! (e.g. derivations)

Extended supports

We would like to define supports in $\mathrm{Lip}_0(M)^*$ in a meaningful way (e.g. the support of $\delta(\xi)$, $\xi \in M^{\mathcal{R}}$ should be $\{\xi\}$).

The **extended support** of $\phi \in \mathrm{Lip}_0(M)^*$ is

$$\mathcal{S}(\phi) = \bigcap \{A \subset M^{\mathcal{U}} \text{ closed} : \phi(f) = 0 \text{ if } f^{\mathcal{U}}\restriction_A = 0\}$$

If $\phi \in \mathcal{F}(M)$ then $\mathcal{S}(\phi) = \overline{\mathrm{supp}(\phi)}^{\mathcal{U}}$ and $\mathrm{supp}(\phi) = \mathcal{S}(\phi) \cap M$.

⚠ It is **not** true that $f^{\mathcal{U}}|_{\mathcal{S}(\phi)} = 0$ implies $\phi(f) = 0$! (e.g. derivations)

Theorem *(Aliaga, Pernecká 2020)*

Suppose that $\phi \in \mathrm{Lip}_0(M)^*$ avoids infinity. Then:

- $\phi(f) = 0$ for any $f \in \mathrm{Lip}_0(M)$ such that $f|_{U\cap M} = 0$ for some open $U \subset M^{\mathcal{U}}$ containing $\mathcal{S}(\phi)$.
- $\mathcal{S}(\phi)$ is the smallest set with that property.

Application 2: Functionals induced by measures

Given a Borel measure μ on M, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_M f \, d\mu$$

Application 2: Functionals induced by measures

Given a Borel measure μ on M, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_M f\,d\mu$$

- We may assume $\mu(\{0\}) = 0$.
- μ may be signed and finite, or positive and σ-finite.

Application 2: Functionals induced by measures

Given a Borel measure μ on M, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_M f\,d\mu$$

Proposition

$\widehat{\mu} \in \mathrm{Lip}_0(M)^*$ iff μ has **finite first moment**, i.e.

$$\int_M d(x,0)\,d\left|\mu\right|(x) < \infty$$

In that case, $\widehat{\mu} \in \mathcal{F}(M)$ and $\mathrm{supp}(\widehat{\mu}) = \mathrm{supp}(\mu)$.

Application 2: Functionals induced by measures

Given a Borel measure μ on $M^{\mathcal{U}}$, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_{M^{\mathcal{U}}} f^{\mathcal{U}}\, d\mu$$

Proposition

$\widehat{\mu} \in \mathrm{Lip}_0(M)^*$ iff μ has **finite first moment**, i.e.

$$\int_M d(x,0)\, d\left|\mu\right|(x) < \infty$$

In that case, $\widehat{\mu} \in \mathcal{F}(M)$ and $\mathrm{supp}(\widehat{\mu}) = \mathrm{supp}(\mu)$.

Application 2: Functionals induced by measures

Given a Borel measure μ on $M^{\mathcal{U}}$, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_{M^{\mathcal{U}}} f^{\mathcal{U}}\, d\mu$$

Proposition

$\widehat{\mu} \in \mathrm{Lip}_0(M)^*$ iff μ has **finite first $\bar{d}$-moment**, i.e.

$$\int_{M^{\mathcal{U}}} \bar{d}(x,0)\, d\,|\mu|\,(x) < \infty$$

In that case, μ is concentrated on $M^{\mathcal{R}}$ and $\mathcal{S}(\widehat{\mu}) = \mathrm{supp}(\mu)$.

Application 2: Functionals induced by measures

Given a Borel measure μ on $M^{\mathcal{U}}$, we consider the functional $\widehat{\mu}$ on $\mathrm{Lip}_0(M)$:

$$\widehat{\mu}(f) = \int_{M^{\mathcal{U}}} f^{\mathcal{U}}\, d\mu$$

Proposition

$\widehat{\mu} \in \mathrm{Lip}_0(M)^*$ iff μ has **finite first $\bar{d}$-moment**, i.e.

$$\int_{M^{\mathcal{U}}} \bar{d}(x,0)\, d\,|\mu|\,(x) < \infty$$

In that case, μ is concentrated on $M^{\mathcal{R}}$ and $\mathcal{S}(\widehat{\mu}) = \mathrm{supp}(\mu)$.

Theorem *(Aliaga, Pernecká 2023)*

If $\widehat{\mu} \in \mathrm{Lip}_0(M)^*$, then $\widehat{\mu} \in \mathcal{F}(M)$ iff μ is concentrated on M.

Riesz meets Lipschitz

A functional ϕ on $\mathrm{Lip}_0(M)$ is **positive** if $\phi(f) \geq 0$ whenever $f \geq 0$ pointwise.

Riesz meets Lipschitz

A functional ϕ on $\mathrm{Lip}_0(M)$ is **positive** if $\phi(f) \geq 0$ whenever $f \geq 0$ pointwise.

Theorem *(Aliaga, Pernecká 2023; Ambrosio, Puglisi 2020)*

For $m \in \mathcal{F}(M)$, TFAE:

1. m is positive,
2. $m = \widehat{\mu}$ where μ is a measure on M satisfying:
   - μ is positive and σ-finite
   - μ has finite first moment
   - $\mu\restriction_A$ is Radon for every closed $0 \notin A \subset M$

Riesz meets Lipschitz, meets Samuel

A functional ϕ on $\mathrm{Lip}_0(M)$ is **positive** if $\phi(f) \geq 0$ whenever $f \geq 0$ pointwise.

Theorem *(Aliaga, Pernecká 2023; Ambrosio, Puglisi 2020)*

For $m \in \mathcal{F}(M)$, TFAE:

1. m is positive,
2. $m = \widehat{\mu}$ where μ is a measure on M satisfying:
   - μ is positive and σ-finite
   - μ has finite first moment
   - $\mu\restriction_A$ is Radon for every closed $0 \notin A \subset M$

Theorem *(Aliaga, Pernecká 2023)*

For $\phi \in \mathrm{Lip}_0(M)^*$, TFAE:

1. ϕ is positive and avoids infinity and the base point,
2. $\phi = \widehat{\mu}$ where μ is a measure on $M^{\mathcal{R}}$ satisfying:
   - μ is positive and σ-finite
   - μ has finite first moment
   - $\mu\restriction_A$ is Radon for every closed $0 \notin A \subset M^{\mathcal{R}}$

Application 3: Optimal de Leeuw representations

A (finite or infinite) sum

$$\sum_{n=1}^{\infty} a_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} \in \mathcal{F}(M)$$

has maximum possible norm $\sum_n a_n$ precisely when $\{(x_n, y_n)\}$ are **cyclically monotonic**, that is

$$d(x_{n_1}, y_{n_1}) + \ldots + d(x_{n_k}, y_{n_k}) \leq d(x_{n_1}, y_{n_2}) + \ldots + d(x_{n_k}, y_{n_1})$$

for all choices of indices $n_1, \ldots, n_k$.
(We call it a *convex sum of molecules*).

Application 3: Optimal de Leeuw representations

A (finite or infinite) sum

$$\sum_{n=1}^{\infty} a_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} \in \mathcal{F}(M)$$

has maximum possible norm $\sum_n a_n$ precisely when $\{(x_n, y_n)\}$ are **cyclically monotonic**, that is

$$d(x_{n_1}, y_{n_1}) + \ldots + d(x_{n_k}, y_{n_k}) \leq d(x_{n_1}, y_{n_2}) + \ldots + d(x_{n_k}, y_{n_1})$$

for all choices of indices $n_1, \ldots, n_k$.
(We call it a *convex sum of molecules*).

Application 3: Optimal de Leeuw representations

A (finite or infinite) sum

$$\sum_{n=1}^{\infty} a_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} \in \mathcal{F}(M)$$

has maximum possible norm $\sum_n a_n$ precisely when $\{(x_n, y_n)\}$ are **cyclically monotonic**, that is

$$d(x_{n_1}, y_{n_1}) + \ldots + d(x_{n_k}, y_{n_k}) \leq d(x_{n_1}, y_{n_2}) + \ldots + d(x_{n_k}, y_{n_1})$$

for all choices of indices $n_1, \ldots, n_k$.
(We call it a *convex sum of molecules*).

More generally, if $\widetilde{M} = \{(x,y) : x \neq y \in M\}$ and μ is a probability measure on $\widetilde{M}$, then

$$\left\| \int_{\widetilde{M}} \frac{\delta(x) - \delta(y)}{d(x,y)} \, d\mu(x,y) \right\|_{\mathcal{F}(M)} = 1$$

precisely when μ is concentrated on a cyclically monotonic set.

Application 3: Optimal de Leeuw representations

Theorem (informal statement) *(Aliaga, Pernecká, Smith 2024)*

If μ is a probability integral on $\widetilde{M^{\mathcal{R}}} = \{(\xi,\zeta) : \xi \neq \zeta \in M^{\mathcal{R}}\}$, then

$$\left\| \int_{\widetilde{M^{\mathcal{R}}}} \frac{\delta(\xi) - \delta(\zeta)}{\bar{d}(\xi,\zeta)} \, d\mu(\xi,\zeta) \right\|_{\mathrm{Lip}_0(M)^*} = 1$$

if and only if μ is concentrated on a $\bar{d}$-cyclically monotonic subset of $M^{\mathcal{R}} \times M^{\mathcal{R}}$.

Thanks for your attention!

References:

- **"Integral representation and supports of functionals on Lipschitz spaces"**, R. J. Aliaga and E. Pernecká, *Int. Math. Res. Not.* 2023 (2023), 3004–3072
- **"Convex integrals of molecules in Lipschitz-free spaces"**, R. J. Aliaga, E. Pernecká and R. J. Smith, *J. Funct. Anal.* (to appear)
- **"de Leeuw representations of functionals on Lipschitz spaces"**, R. J. Aliaga, E. Pernecká and R. J. Smith, *preprint available on arXiv*
- **"The Samuel realcompactification of a metric space"**, M. I. Garrido and A. S. Meroño, *J. Math. Anal. Appl.* 456 (2017), 1013–1039
- **"Lipschitz algebras"**, 2nd. edition, N. Weaver, 2018
- **"The minimum uniform compactification of a metric space"**, R. G. Woods, *Fund. Math.* 147 (1995), 39–59