Banach lattices

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Proyecto PID2021-122126NB-C32 financiado por MICIU/AEI /10.13039/501100011033/ y por FEDER Una manera de hacer Europa y 21955/PI/22 by Fundación Séneca, ACyT Región de Murcia

Castro Urdiales 2024

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- Any inequality that is true in $\mathbb{R}$ is true in X
- For $|x| := x \vee -x$,

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

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- A "Gurarii" Banach lattice", A "Bossard" descriptive set theory of Banach lattices
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Convergence notions in Banach lattices

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They do not come from a topology

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Are the coordinate functionals continuous?

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Problem (Gumenchuk, Karlova, Popov / Taylor, Troitsky)

Are the coordinate functionals continuous?

What follows is joint work with C. Rosendal, M. Taylor and P. Tradacete.

Topological group approach

Coordinate functionals define a group homomorphism

$$\begin{array}{ccc} X & \xrightarrow{E} & \mathbb{R}^{\mathbb{N}} \\ x & \mapsto & (a_k(x))_k \end{array}$$

Theorem (Pettis)

A group homomorphism between Polish groups is continuous if and only if it is Baire measurable.

Theorem

If the graph of E is analytic then E is Baire measurable.

We have to analyze the complexity of the graph $\{(x, y) : Ex = y)\}$.

$$E(x) = y \quad \Leftrightarrow \quad x = \lim{}^{c} \sum_{k=1}^{n} y_k e_k$$

u-bases have continuous functionals

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One exists over Polish, all other quantifiers over $\mathbb{N}$, the graph is analytic and the coordinates continuous!

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$$\exists (z_m) \in X \quad \inf\{z_m\} = 0 \text{ and } \forall m \quad \exists N \quad \forall n > N \quad \left| x - \sum_{k=1}^{n} y_k e_k \right| \leq z_m$$

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The problem here comes from the infimum condition, that produces an extra quantifier $\forall w \in$ Polish

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So now the graph is Σ^1_2. Under Σ^1_1-determinacy or $MA \neg CH$ this implies that E Baire measurable and then again continuous.

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Theorem

$$\left\{((z_m)_m, X) : (z_m) \in X_+^{\mathbb{N}},\ \inf\{z_m\} = 0\right\}$$

is coanalytic not Borel.

Here X varies in the space of separable Banach lattices, similar to Bossard theory, studied by Tursi.

Complexity of order null sequences

The set

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Indeed $ONS(X)$ is Borel if and only if X is α-Fatou for some $\alpha < \omega_1$.

The α-Fatou game

Fix a decreasing sequence $(z_m) \subset X_+$

Player I	α_1	$>$	α_2	$>$	α_3	$>$	$\cdots$		*ordinals* $< \alpha$
Player II		y_1		y_2		y_3	$\cdots$		*vectors* $\in X_+$

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Player II tries closer and closer approximate lower bounds.

$$\|(y_i - z_n)^+\| < \frac{\|y_i\|}{2^i}$$

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First player that cannot move loses.

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X is α-Fatou if $\inf(z_m) = 0 \Leftrightarrow$ Player II does not have winning strategy in the α-Fatou game.

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We know that there are Banach lattices arbitrarily high in the hierarchy... but we do not know if there are Banach lattices that are not α-Fatou for any $\alpha < \omega_1$.

Failing Fatou properties

- The space c of convergent sequences with the norm

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In order to fail the α-Fatou one transfinitely iterates this idea, with some techinicalities.

Solving some questions of Taylor and Troitsky

Theorem

Suppose that $\overline{span}\{e_n\} = X$, *and* e_n^* *are biorthogonal.*

- e_n *is* σo*-basis of* X *with coordinates* $e_n^* \Rightarrow$ *(*Σ_1^1*-determiacy)*
- e_n *is u-basis of* X *with coordinates* $e_n^* \Rightarrow$
- e_n *is Schauder basis of* X *with coordinates* e_n^*

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Theorem

Suppose that $\overline{span}\{e_n\} = X$*, TFAE*

1. e_n *is u-basis of* X
2. *There is a constant* M *such that for all scalars* $a_1, \dots, a_m$*,*

$$\left\| \bigvee_{k=1}^{m} \left| \sum_{n=1}^{k} a_n e_n \right| \right\| \leq M \left\| \sum_{n=1}^{m} a_n e_n \right\|$$