Banach lattices

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- Any inequality that is true in $\mathbb{R}$ is true in X
- For $|x| := x \vee -x$,

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

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- c_0, ℓ_p, $C(K)$, $L_p(\mu)$...

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- A "Gurarii" Banach lattice", A "Bossard" descriptive set theory of Banach lattices
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Convergence notions in Banach lattices

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They do not come from a topology

Basis notions in Banach lattices

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Are the coordinate functionals continuous?

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Are the coordinate functionals continuous?

What follows is joint work with C. Rosendal, M. Taylor and P. Tradacete.

Topological group approach

Coordinate functionals define a group homomorphism

$$\begin{array}{ccc} X & \xrightarrow{E} & \mathbb{R}^{\mathbb{N}} \\ x & \mapsto & (a_k(x))_k \end{array}$$

Theorem (Pettis)

A group homomorphism between Polish groups is continuous if and only if it is Baire measurable.

Theorem

If the graph of E is analytic then E is Baire measurable.

We have to analyze the complexity of the graph $\{(x,y) : Ex = y)\}$.

$$E(x) = y \quad \Leftrightarrow \quad x = \lim{}^{c} \sum_{k=1}^{n} y_k e_k$$

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One exists over Polish, all other quantifiers over $\mathbb{N}$, the graph is analytic and the coordinates continuous!

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So now the graph is Σ^1_2. Under Σ^1_1-determinacy or $MA\neg CH$ this implies that E Baire measurable and then again continuous.

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Theorem

$$\left\{((z_m)_m, X) : (z_m) \in X_+^{\mathbb{N}},\ \inf\{z_m\} = 0\right\}$$

is coanalytic not Borel.

Here X varies in the space of separable Banach lattices, similar to Bossard theory, studied by Tursi.

Complexity of order null sequences

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Indeed $ONS(X)$ is Borel if and only if X is α-Fatou for some $\alpha < \omega_1$.

The α-Fatou game

Fix a decreasing sequence $(z_m) \subset X_+$

Player I	α_1	$>$	α_2	$>$	α_3	$>$	$\cdots$	*ordinals* $< \alpha$
Player II		y_1		y_2		y_3	$\cdots$	*vectors* $\in X_+$

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Player II tries closer and closer approximate lower bounds.

$$\|(y_i - z_n)^+\| < \frac{\|y_i\|}{2^i}$$

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First player that cannot move loses.

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X is α-Fatou if $\inf(z_m) = 0 \Leftrightarrow$ Player II does not have winning strategy in the α-Fatou game.

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We know that there are Banach lattices arbitrarily high in the hierarchy... but we do not know if there are Banach lattices that are not α-Fatou for any $\alpha < \omega_1$.

Failing Fatou properties

- The space c of convergent sequences with the norm

$$\|x\|' = \frac{1}{2}\|x\|_\infty \vee \lim_n |x_n|$$

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In order to fail the α-Fatou one transfinitely iterates this idea, with some techinicalities.

Solving some questions of Taylor and Troitsky

Theorem

Suppose that $\overline{span}\{e_n\} = X$, and e_n^* are biorthogonal.

- e_n *is* σo*-basis of* X *with coordinates* e_n^* $\Rightarrow$ *(*Σ_1^1*-determiacy)*
- e_n *is u-basis of* X *with coordinates* e_n^* $\Rightarrow$
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Theorem

Suppose that $\overline{span}\{e_n\} = X$, *TFAE*

1. e_n *is u-basis of* X
2. *There is a constant* M *such that for all scalars* $a_1, \ldots, a_m$,

$$\left\| \bigvee_{k=1}^{m} \left| \sum_{n=1}^{k} a_n e_n \right| \right\| \leq M \left\| \sum_{n=1}^{m} a_n e_n \right\|$$

Part III

Projective Banach spaces

Theorem

For a Banach space X TFAE

1. Each operator $Y \longrightarrow X$ onto X has continuous linear selection

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The lifting property

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4. ~~X is isomorphic to $FBL[\ell_1(\Gamma)]$~~

Open problem

We do not know any example of X as above where we cannot take

$$\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$$

¿ projective $\Leftrightarrow$ 1^+-projective ?

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Open problem

Is there a projective $FBL[E]$ with $E \ncong \ell_1(\Gamma)$?

E must have the Schur property. (A., Martínez Cervantes, Rodríguez Abellán)

Embeddings of $C[0,1]$

Projectivity was one of the ingredients in the proof of:

Theorem (A., Martínez Cervantes, Rueda Zoca, Tradacete)

If a Banach lattice X contains a subspace $C[0,1]$, then X also contains a sublattice $C[0,1]$.

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Open problem

If a Banach lattice X contains a subspace $C([0,1]^\kappa)$, does X also contain a sublattice $C([0,1]^\kappa)$?

If $X = C(K)$, true for $\kappa = \mathfrak{c}^+$, for cardinals with MA_κ and other cases. Plebanek, Haydon, Fremlin...

Injective Banach spaces

Theorem (Goodner, Nachbin)

For a Banach space X TFAE

1. If $Z \subset Y$, every operator $T : Z \longrightarrow X$ has an extension $\tilde{T} : Y \longrightarrow X$, with $\|\tilde{T}\| = \|T\|$.

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Famous open problem

Is every injective space isomorphic to a 1-injective space?

Injective Banach lattices

Theorem (de Pagter, Wickstead)

There are no injective Banach lattices.

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If X was injective, we could extend $\mathit{Constants} \longrightarrow X$ to some $L^1([0,1]^\kappa) \longrightarrow X$ for arbitrarily large κ, impossible.

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Theorem (Sobczyk, Zippin)

c_0 is the only separable separably injective Banach space.
(If $Z \subset Y$ are separable, $Z \longrightarrow c_0$ extends to $Y \longrightarrow c_0$.)

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c_0 is the only separable separably injective Banach space.
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Theorem (A., Tradacete)

- There is no separable separably injective Banach lattice.
- $L_1([0,1]^{\omega_1})$ is 1-separably injective

Amalgamation property and Gurarii-like construction

Theorem (A., Tradacete // Tursi)

If we have isometric lattice embeddings

$$\begin{array}{ccc} A & & \\ \uparrow & & \\ R & \rightarrow & B \end{array}$$

Then there are isometric lattice embeddings

$$\begin{array}{ccc} A & \color{red}{\rightarrow} & \color{red}{P} \\ \uparrow & & \color{red}{\uparrow} \\ R & \rightarrow & B \end{array}$$

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Our motivation was to build
lattices of universal disposition for separable lattices.
Her motivation was to build Gurarii-like
lattices of almost universal disposition for finite dimensional lattices

The complemented subspace problem

Theorem (de Hevia, Martínez Cervantes, Salguero Alarcón, Tradacete // after Plebanek, Salguero Alarcón)

There exists a complemented subspace of a Banach lattice that does not have a Banach lattice structure.

The complemented subspace problem

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Open problem

Is there a separable example?

WCG and weakly null absolute values

A Banach space E is WCG if $E = \overline{span}(K)$ with K weakly compact.

Open problem

Characterize for which E is $FBL[E]$ WCG.

WCG and weakly null absolute values

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Open problem

Characterize for which E is $FBL[E]$ WCG.

$FBL[\ell_p(\Gamma)]$ is WCG if and only if $p > 2$.

(A., Rodríguez, Tradacete // A., Tradacete, Villanueva)

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Related open problem

Characterize when a weakly null sequence in E has weakly null absolute values in $FBL[E]$ ($=$ in any Banach lattice where E sits).

I would like to blame the organizers

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for their attempts at confusing me

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for their attempts at confusing me and even at my physical elimination

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Satisfaction expression feeling that my end is near