Ball separation charesterization of small diameter properties

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Introduction

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Definition

- Let $x^* \in X^*$, $\alpha > 0$ and $C \subseteq X$. Then the set $S(C, x^*, \alpha) = \{x \in C : x^*(x) > \sup x^*(C) - \alpha\}$ is called the open slice determined by x^* and α. One can analogously define w^* slices $S(D, x, \alpha) = \{x^* \in D : x^*(x) > \sup x(D) - \alpha\}$ in X^*.

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- Let $o \leq \lambda \leq 1$ and S_i 's are slices of C. We define Small Combination of Slices(SCS) = $\sum_{i=1}^{n} \lambda_i S_i$

$x^* = \sup_{y \in B} x^*(y)$ $\qquad$ $x^* = \sup_{y \in B} x^*(y) - \alpha$

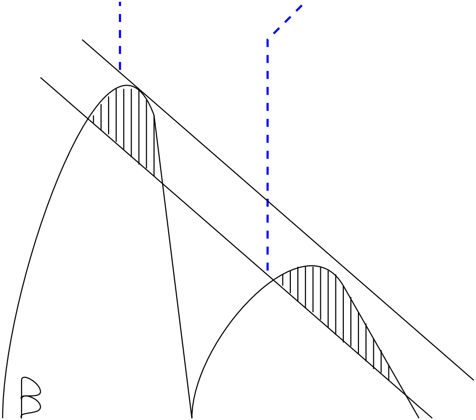

Figure: slice

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- Bourgain in his work "La propriÃ©tÃ© de Radon-Nikodym" (1979) first mentioned the concept of SCS which later on became famous Bourgain's Lemma. He also introduced a "strongly regular" set namely a nonempty convex set with small SCS i.e. with arbitrarily small diameter.

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- Later N.Ghoussoub , G.Godefroy , B. Maurey, W. Scachermayer in their monograph, "Some topological and geometrical structures in Banach spcaes", (1987), addressed these three aspects in details. .

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- GGMS(1987) X is Strongly Regular (*SR*) iff every closed, convex and bounded subset of X has SCS with arbitrarily small diameter.
- A slice is a weakly open set so, $RNP \Rightarrow PCP$. Also Bourgain's Lemma tells us any weakly open set in a closed, bounded and convex set contains a SCS, so $PCP \Rightarrow SR$. It is also well known that none of these implications can be reversed.

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Remark

Analogously we can define w^*-*BSCSP*, w^*-*BHP* and w^*-*BDP* in a dual space.

Small Diameter Properties

Clearly, *BDP* always implies *BHP*, in fact, any slice of the unit ball is relatively weakly open. *BHP* implies *BSCSP* follows from Bourgain's Lemma, which says that every non-empty relatively weakly open subset of B_X contains a finite convex combination of slices. Similar observations are true for w^*-versions. Since every w^*-slice (w^*-open set) of B_{X^*} is also a slice (weakly open set) of B_{X^*}, so we have the following diagram :

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$$\begin{array}{ccccc} BDP & \Longrightarrow & BHP & \Longrightarrow & BSCSP \\ \Uparrow & & \Uparrow & & \Uparrow \\ w^*BDP & \Longrightarrow & w^*BHP & \Longrightarrow & w^*BSCSP \end{array}$$

In general, none of the reverse implications of the diagram hold.

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- Schaermeyer (1987) later proved X has *RNP* if and only if it is SR and has the Krein Milman Property (KMP),
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- Ball separation charecterization (Basu ad Seal 2024)

The spaces that we will be considering have been well studied in literature. A large class of function spaces like the Bloch spaces, Lorentz and Orlicz spaces, spaces of vector valued functions and spaces of compact operators are some examples.

Introduction to Ball separation property

- It is a well-known fact that if we have a closed bounded convex set C in a Banach space X and a point $x \notin C$, then we will get a Half-space H such that $C \subset H$ and $x \notin H$.

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- One may ask whether the same kind of separation can be achieved by closed balls.

Some well known Ball Separation Properties

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3. (Chen and Lin, 1998) Property (II) : If for every closed bounded convex set C in X and a point $x \notin C$, there exist closed balls $B_1, \ldots, B_n$ in X such that $C \subset \overline{co}(\bigcup_{i=1}^{n} B_i)$ and $x \notin \overline{co}(\bigcup_{i=1}^{n} B_i)$.

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4. In their paper, Chen and Lin charesterized several other well known geometric properties of Banach spaces in terms of Ball separation.

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Then $(1) \implies (2) \iff (3) \iff (4)$

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   for every bounded set C in X^* with $\inf x_0(C) > \varepsilon$, then there exist balls $B_1, B_2, \ldots, B_n$ in X^* such that $C \subset \bigcup_{i=1}^{n} B_i$ and $0 \notin \bigcup_{i=1}^{n} B_i$.
3. Given $\varepsilon > 0$, there exists a hyperplane $H = \{x^* \in X^* : x^*(x_0) = 0\}$ with $x_0 \in B_X$ such that
   for every bounded set C in X^* with $d(C, H) > \varepsilon$, then there exists a family $\{T_i : i \in I\}$, where each T_i is a finite union of balls in X^*, such that $C \subset \bigcap_{i \in I} T_i$ and $(\bigcap_{i \in I} T_i) \bigcap H = \emptyset$.

Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4)

$\mathcal{A}$-SCS point

- Let C be a bounded convex subset in X. A point $x \in C$ is called a Small Combination of Slices point (SCS point) of C if for every $\varepsilon > 0$, there exists convex combination of slices $T = \sum_{i=1}^{n} \lambda_i S_i$ of C such that $x \in T$ and diameter of T is less than ε. Analogously one can define w^*-Small Combination of Slices (w^*-SCS) point in X^*.
- (Chen and Lin 1998) A collection $\mathcal{A}$ of bounded subsets of X is said to be compatible if it satisfies the followings :
  1. If $A \in \mathcal{A}$ and $C \subset A$, then $C \in \mathcal{A}$.
  2. For each $A \in \mathcal{A}$, $x \in X$, $A + x \in \mathcal{A}$ and $A \bigcup \{x\} \in \mathcal{A}$.
  3. For each $A \in \mathcal{A}$, the closed absolutely convex hull of A is in $\mathcal{A}$.

$\mathcal{A}$-SCS point

- Let $\mathcal{A}$ be a collection of bounded subset in X. Then $f \in X^*$ is said to be a $\mathcal{A}$-Small Combination of Slice ($\mathcal{A}$-SCS) point of B_{X^*} if for each $A \in \mathcal{A}$ and $\varepsilon > 0$ there exists a convex combination of w^*-slices $T = \sum_{i=1}^{n} \lambda_i S_i$ in B_{X^*} such that $f \in T$ and $\text{diam}_A(T) < \epsilon$.
- If we take $\mathcal{A}$ as all bounded subsets of X then the $\mathcal{A}$-SCS Point of B_{X^*} is essentially the w^*-SCS point of B_{X^*}.

$\mathcal{A}$-SCS point and its ball separation characterizaton

- Let X be a Banach space and f_0 be an $\mathcal{A}$-SCS point of B_{X^*}. Then for all $A \in \mathcal{A}$, if $\inf f_0(A) > 0$, then there exist balls $B_1, B_2, \ldots, B_n$ in X such that $A \subset \bigcup_{i=1}^{n} B_i$ and $0 \notin \bigcup_{i=1}^{n} B_i$.
- Let X be a Banach space and $f_0 \in B_{X^*}$ is a w^*-SCS point of B_{X^*}. Then the following equivalent conditions are true.
  1. For every bounded set C in X with $\inf f_0(C) > 0$, then there exist balls $B_1, B_2, \ldots, B_n$ in X such that $C \subset \bigcup_{i=1}^{n} B_i$ and $0 \notin \bigcup_{i=1}^{n} B_i$.
  2. For every bounded set C in X^{**} with $\inf f_0(C) > 0$, then there exist balls $B_1, B_2, \ldots, B_n$ in X^{**} with center in X such that $C \subset \bigcup_{i=1}^{n} B_i$ and $0 \notin \bigcup_{i=1}^{n} B_i$.
  3. Let $H = \{x \in X : f_0(x) = 0\}$. Then for every bounded set C in X with $d(C, H) > 0$, there exists a family $\{T_i : i \in I\}$, where each T_i is a finite union of balls in X, such that $C \subset \bigcap_{i \in I} T_i$ and $(\bigcap_{i \in I} T_i) \bigcap H = \emptyset$.

SCS points and its ball separation characterizaton

Let X be a Banach space and $x_0 \in B_X$ is a SCS point of B_X. Then the following equivalent conditions are true.

1. For every bounded set C in X^* with $\inf x_0(C) > 0$, then there exist balls $B_1, B_2, \ldots, B_n$ in X^* such that $C \subset \bigcup_{i=1}^{n} B_i$ and $0 \notin \bigcup_{i=1}^{n} B_i$.
2. Let $H = \{x^* \in X^* : x^*(x_0) = 0\}$. Then for every bounded set C in X^* with $d(C, H) > 0$, there exists a family $\{T_i : i \in I\}$, where each T_i is a finite union of balls in X^*, such that $C \subset \bigcap_{i \in I} T_i$ and $(\bigcap_{i \in I} T_i) \bigcap H = \emptyset$.

SCS points and its ball separation characterizaton

- Let X be a Banach space and every point in B_X be SCS point of B_X. Then for every bounded set C in X and any w^*-closed hyperplane H in X^*, if $d(C, H) > 0$, then there exists a family $\{T_i : i \in I\}$, where each T_i is a finite union of balls in X^*, such that $C \subset \bigcap_{i \in I} T_i$ and $(\bigcap_{i \in I} T_i) \bigcap H = \emptyset$.
- Let X be a Banach space and every point in B_X be SCS point of B_X. Then for every bounded set C in X and any w^*-closed hyperplane H in X^*, if $d(C, H) > 0$, then there exists a family $\{T_i : i \in I\}$, where each T_i is a finite union of balls in X^*, such that $C \subset \bigcap_{i \in I} T_i$ and $(\bigcap_{i \in I} T_i) \bigcap H = \emptyset$.

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Thank You!