Trimming the Johnson bonsai

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New Perspectives in Banach spaces
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Joint work with Félix Cabello and Yolanda Moreno

Every Banach space is a quotient of some $\ell_1(\Gamma)$

- How?
- In how many ways?
- In how many **different** ways?
- Projective spaces

Banach *p*-**Banach** **Quasi-Banach**

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$\ell_1(\Gamma)$	$\ell_p(\Gamma)$	— — — — — — — — —

Every complemented subspace of $\ell_p(\Gamma)$ is isomorphic to some $\ell_p(\Gamma')$

$$0 < p < \infty$$

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KÖTHE, G.
Math. Annalen 165, 181—195 (1966)

Hebbare lokalkonvexe Räume

ERNST HÖLDER zum 65. Geburtstag

GOTTFRIED KÖTHE

1. Einleitung

Seit einer Arbeit von PHILLIPS [13] hat man sich mit der Frage der Fortsetzbarkeit stetiger linearer Abbildungen von Banachräumen beschäftigt. NACHBIN [11] gab kürzlich einen Überblick über die bisherigen Resultate und wies darauf hin, daß das duale Problem des Hochhebens von Abbildungen noch wenig bearbeitet ist. Wir wollen uns hier mit diesem dualen Problem beschäftigen.

Zum Verständnis der Fragestellung seien kurz einige Begriffe und Resultate der Fortsetzungstheorie erwähnt. Ein Banachraum E heißt ein $\mathfrak{P}_\lambda$-Raum, $\lambda \geqq 1$, wenn jeder (B)-Raum X, der einen zu E normisomorphen Teilraum H enthält, eine stetige Projektion P auf H mit $\|P\| \leqq \lambda$ besitzt. Der Zusammenhang mit der stetigen Fortsetzbarkeit ist der folgende: H sei ein zu E normisomorpher Teilraum des (B)-Raumes X, A sei eine stetige lineare Abbildung von H in den (B)-Raum Y; dann existiert eine Fortsetzung B von A, die X stetig in Y abbildet und es ist $\|B\| \leqq \lambda \|A\|$. Diese Aussage ist äquivalent damit, daß E ein $\mathfrak{P}_\lambda$-Raum ist.

Völlig geklärt ist bis heute nur die Struktur der $\mathfrak{P}_1$-Räume. Jeder $\mathfrak{P}_1$-

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- Everybody knows how to make the separable case.
- Köthe did the $\ell_1(\Gamma)$ case.
- Ortyński did the $p < 1$ case.

Every complemented subspace of $\ell_p(\Gamma)$ is isomorphic to some $\ell_p(\Gamma')$

BULLETIN DE L'ACADÉMIE POLONAISE DES SCIENCES
Série des sciences math., astr. et phys. — Vol. XXVI, No. 1, 1978

MATHEMATICS
(FUNCTIONAL ANALYSIS)

On Complemented Subspaces of $l^p(\Gamma)$ for $0<p\leqslant 1$

by

Augustyn ORTYŃSKI

Presented by W. ORLICZ on May, 4, 1977

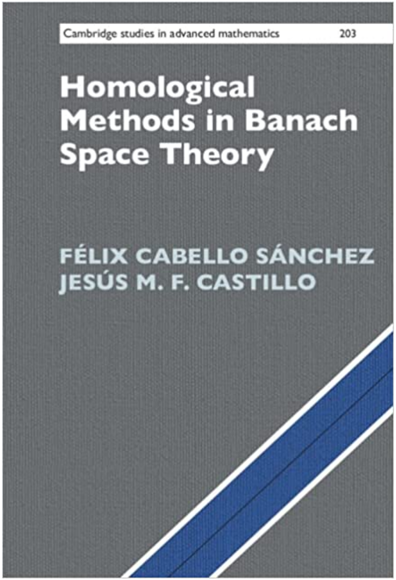
Summary. It is proved that each complemented subspace of $l^p(\Gamma)$ for $0<p\leqslant 1$ is isomorphic to $l^p(\Gamma')$ for some $\Gamma'\subset\Gamma$. This implies that $l^p(\Gamma)$'s are the only projective spaces in the class of all p-Banach spaces.

Given a set Γ with card $\Gamma=\mathfrak{m}$ and $0<p<\infty$, $l^p_{\mathfrak{m}}=l^p(\Gamma)$ denotes the space of all scalar-valued functions x defined on Γ such that $\|x\|_p=(\sum_{\gamma\in\Gamma}|x(\gamma)|^p)^{1/r}<\infty$, where $r=\max\{1,p\}$. $(l^p(\Gamma), \|\cdot\|_p)$ is a Banach space for $p\geqslant 1$, and a p-Banach space for $0<p<1$. (A p-Banach space is metrizable complete topological linear space whose topology may be defined by a p-norm, i.e. a p-homogeneous norm, cf. [5]). We write shortly l^p when $\mathfrak{m}=\aleph_0$.

Note that, for $0<p<1$, $l^p(\Gamma)$ embeds continuously and densely in $l^1(\Gamma)$ and the dual spaces of $l^p(\Gamma)$ and $l^1(\Gamma)$ may be identified in a natural manner.

In [4] Pełczyński proved that each infinite dimensional complemented subspace of l^p, $p\geqslant 1$, is isomorphic to l^p. Stiles [9] extended this result to the spaces l^p, $0<p<1$. On the other hand, Köthe [2] showed that every complemented subspace of $l^1(\Gamma)$ is isomorphic to $l^1(\Gamma')$ for some $\Gamma'\subset\Gamma$ (see also the first Corollary on p. 29 in [7] and [6]).

The book


Cambridge studies in advanced mathematics 203
Homological Methods in Banach Space Theory
FÉLIX CABELLO SÁNCHEZ
JESÚS M. F. CASTILLO

Decompositions

- Every subspace H of $c_0(\Gamma)$ has the form $\mathrm{H} = \mathrm{c}_0(\Gamma, X_\alpha)$ with $X_\alpha \subset c_0$

Y. Moreno, A. Plichko, *On automorphic Banach spaces*, Israel J. Math. 169 (2009) 29–45.

ISRAEL JOURNAL OF MATHEMATICS **169** (2009), 29–45
DOI: 10.1007/s11856-009-0002-4

ON AUTOMORPHIC BANACH SPACES

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Decompositions

- Every subspace H of $c_0(\Gamma)$ has the form $\mathrm{H} = \mathrm{c}_0(\Gamma, X_\alpha)$ with $X_\alpha \subset c_0$
- Every subspace H of $\ell_p(\Gamma)$, $p > 1$ has the form $\mathrm{H} = \ell_p(\Gamma, X_\alpha)$ with $X_\alpha \subset \ell_p$

Y. Moreno, A. Plichko, *On automorphic Banach spaces*, Israel J. Math. 169 (2009) 29–45

$+$

W.B. Johnson, M. Zippin, *Extension of operators from subspaces of $c_0(\Gamma)$ into $C(K)$ spaces*, Proc. Amer. Math. Soc. 107 (1989) 751–754.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 107, Number 3, November 1989

EXTENSION OF OPERATORS FROM SUBSPACES OF $c_0(\Gamma)$ INTO $C(K)$ SPACES

W. B. JOHNSON AND M. ZIPPIN

(Communicated by William J. Davis)

ABSTRACT. It is shown that for every $\varepsilon > 0$, every bounded linear operator T from a subspace X of $c_0(\Gamma)$ into a $C(K)$ space has an extension $\mathbf{T}$ from $c_0(\Gamma)$ into the $C(K)$ space such that $\|\mathbf{T}\| \leq (1+\varepsilon)\|T\|$. Even when Γ is countable, T is compact, and X has codimension 1 in c_0, the "ε" cannot be replaced by 0. These results answer questions raised by J. Lindenstrauss and A. Pełczynski in 1971.

J. Lindenstrauss and A. Pełczynski proved [LP, Theorem 3.1] that for every $\varepsilon > 0$, every operator T from a subspace of c_0 into a $C(K)$ space has an extension $\mathbf{T}$ from c_0 into the $C(K)$ space such that $\|\mathbf{T}\| \leq (1+\varepsilon)\|T\|$. They

Decompositions

- Every subspace H of $c_0(\Gamma)$ has the form $\mathrm{H} = c_0(\Gamma, X_\alpha)$ with $X_\alpha \subset c_0$
- Every subspace H of $\ell_p(\Gamma)$, $p > 1$ has the form $\mathrm{H} = \ell_p(\Gamma, X_\alpha)$ with $X_\alpha \subset \ell_p$
- Not every subspace H of $\ell_p(\Gamma)$, $p \leq 1$ has the form $\mathrm{H} = \ell_p(\Gamma, X_\alpha)$ with $X_\alpha \subset \ell_p$.

Think about a quotient $\ell_1(\mathfrak{c}) \longrightarrow \ell_\infty$

Decompositions

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- Not every subspace H of $\ell_p(\Gamma)$, $p \leq 1$ has the form $\mathrm{H} = \ell_p(\Gamma, X_\alpha)$ with $X_\alpha \subset \ell_p$.
- The kernel of the quotient map

$$\ell_1(\mathfrak{m}) \longrightarrow L_1(2^{\mathfrak{m}})$$

is not even isomorphic to a space of the form $\ell_p(\Gamma, X_\alpha)$ with $X_\alpha \subset \ell_1$.

How good (or bad) can a subspace of $\ell_1(\Gamma)$ be?

Somehow this is the heart of Homology

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Think for instance about ℓ_∞/c_0

One knows everything there is to know

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty/c_0 \longrightarrow 0$$

or

$$0 \longrightarrow C_0(\mathbb{N}) \longrightarrow C(\beta\mathbb{N}) \longrightarrow C(\beta\mathbb{N} \setminus \mathbb{N}) \longrightarrow 0$$

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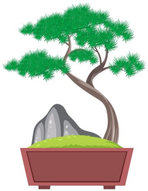
$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
0 & \longrightarrow & C_0(\mathbb{N}) & \longrightarrow & C(\beta\mathbb{N}) & \longrightarrow & C(\beta\mathbb{N}\setminus\mathbb{N}) & \longrightarrow & 0
\end{array}$$

How good (or bad) can a subspace of $\ell_1(\Gamma)$ be?

- Think about

$$0 \longrightarrow \ker Q \longrightarrow \ell_1(m) \xrightarrow{Q} X \longrightarrow 0$$

- Try to think categorically.
- Can we create a "functorial" kernel $\ker Q$?
- Yes, we can, we will call it $co_1(X)$



How good (or bad) can a subspace of $\ell_1(\Gamma)$ be?

- A Banach space X is said to have the Separable Complementation Property (SCP) if every separable subspace is contained in a separable subspace complemented in X.

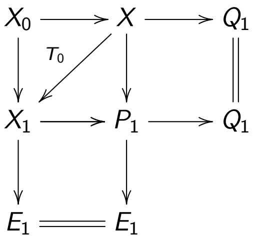
- A Banach space X is said to have the Separable Extension Property (SEP) if for every separable subspace E there is an operator $T : X \to X$ with separable range and such that $T|_E = id$.

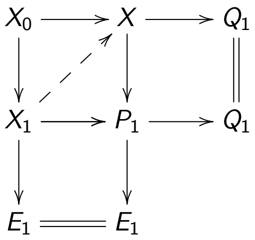
How good (or bad) can a subspace of $\ell_1(\Gamma)$ be?

Theorem

A Banach space with dimension $\aleph_1$ has the SEP if and only if it is a complemented subspace of a space with SCP

Everything you need is contained in two diagrams:





$$P_1 \simeq X_1 \oplus Q_1 \qquad\qquad P_1 \simeq X \oplus E_1$$

.

In other words, $X_0 \subset P_1$ is contained in a separable complemented subspace X_1 of P_1, and X is also complemented in P_1.

How good (or bad) can a kernel be?

Theorem
If X has SCP

$$0 \longrightarrow \ker Q \longrightarrow \ell_1(m) \xrightarrow{Q} X \longrightarrow 0$$

then $\ker Q$ *has SEP.*