Isometries of Lipschitz-free Banach spaces

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Given a Banach space X, describe all isometries (the group of isometries) of X.

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Recall:

Mazur-Ulam theorem

Let X be a real Banach space and $f : X \to X$ a surjective isometry. Then f is affine, i.e. a composition of a linear isometry with a translation.

Conclusion

We may safely focus just on linear isometries.

All linear isometries in this talk will be surjective!

Isometries of classical Banach spaces

Banach-Stone theorem

Let K be a compact Hausdorff space. Then every linear isometry T of $C(K)$ is of the form

$$T(f)(x) = h(x)f\big(\phi(x)\big),\ f \in C(K), x \in K,$$

where $h : K \to \{-1, 1\}$ is continuous and $\phi : K \to K$ is a homeomorphism.

One direction is clear. That is, the content of the theorem is that there are no "surprising" linear isometries.

Let (X, Σ, μ) be a measure space and let $T : X \to X$ be a measurable bijection. Denote by $T^*\mu$ the pushforward of the measure μ by T. T is called *measure preserving* if $T^*\mu = \mu$ and it is called *measure class preserving* if $\mu \sim T^*\mu$, i.e. μ and $T^*\mu$ share the same null-sets.

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If μ is σ-finite and T is measure class preserving, then we can apply the Radon-Nikodym theorem to obtain a non-negative valued function $\frac{dT^*\mu}{d\mu}$ such that for every $A \in \Sigma$ we have

$$T^*\mu(A) = \mu\big(T^{-1}(A)\big) = \int_A \frac{dT^*\mu}{d\mu} d\mu.$$

Isometries of classical Banach spaces

Banach-Lamperti theorem

Let (X, Σ, μ) be a σ-finite measure space and let $p \in [1, \infty) \setminus \{2\}$. Then every linear isometry T of $L_p(\mu)$ is of the form

$$T(f)(x) = h(x) f\big(\phi(x)\big)\big(\frac{d\phi^*\mu}{d\mu}(x)\big)^{1/p},\ f \in L_p(\mu), x \in X,$$

where h is measurable and a.e. 1 or -1, and $\phi : X \to X$ is measure class preserving.

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There has been an extensive research on isometries of other Banach spaces. We refer to monographs *R.J. Fleming and J.E. Jamison*, **Isometries on Banach spaces**, volume I and II.

Lipschitz-free Banach spaces

Let X be a metric space. Let $\mathrm{Mol}(X)$ denote the vector space of all *molecules* over X, i.e. the space

$$\{m : X \to \mathbb{R} \colon \sum_{x \in X} m(x) = 0,\ m \text{ finitely supoorted}\}.$$

Consider the norm on $\mathrm{Mol}(X)$ defined by, for $m \in \mathrm{Mol}(X)$, $\|m\| := \sup_{f \in B_{\mathrm{Lip}(X)}} \langle m, f \rangle$, where

$$\langle m, f \rangle := \Big| \sum_{x \in X} m(x) f(x) \Big|.$$

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Æ(X) is defined as the completion of $\mathrm{Mol}(X)$ with respect to this norm. Recall that $\mathcal{F}(X)$ is defined as the closed linear span of $\{\delta(x) \colon x \in X \setminus \{0\}\} \subseteq \mathrm{Lip}_0^*(X)$, however Æ$(X) \equiv \mathcal{F}(X)$.

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The advantage of the definition of Æ(X) is that it is immediately clear that any surjective isometry of X (not necessarily preserving any distinguished point) extends to a surjective linear isometry of Æ(X).

Lipschitz-free Banach spaces

For a metric space M and $x \neq y \in M$ denote by

- $m_{x,y} := \delta(x) - \delta(y) \in \mathcal{F}(M)$ the elementary molecule (obtained from x and y);
- $n_{x,y} := \frac{m_{x,y}}{d(x,y)} \in \mathcal{F}(M)$ the *normalized* elementary molecule.

Lipschitz-free Banach spaces

Our project

To describe isometries of (a large class of) Lipschitz-free Banach spaces.

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Recall that a map $f : M \to N$ between metric spaces is called a *dilation* if there is a positive constant $K_f > 0$ such that for all $x, y \in M$ we have

$$d_N\big(f(x), f(y)\big) = K_f d_M(x, y).$$

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Definition

Call a metric space X *Lipschitz-free rigid* (or just rigid if there is no risk of misunderstanding) if every linear isometry of Æ(X) is of the form $\pm T_f$, where $f : M \to M$ is a surjective dilation and T_f is defined by

$$T_f(m_{x,y}) = \frac{m_{f(x),f(y)}}{K_f}.$$

Lipschitz-free rigid metric spaces

Main task

Which metric spaces are Lipschitz-free rigid?

Lipschitz-free rigid metric spaces

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Which metric spaces are Lipschitz-free rigid?

Example

$\mathbb{N}$ and $\mathbb{R}$ are not Lipschitz-free rigid.

Lipschitz-free rigid metric spaces

A metric space X is *concave* if every non-trivial triangle inequality is strict and it is *uniformly concave* if for every $x \neq y \in X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in X$ satisfying $\min\{d(x,z), d(y,z)\} \geq \varepsilon$ we have $d(x,y) < d(x,z) + d(z,y) - \delta$.

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Theorem (Mayer-Wolf (1981)/ Weaver (1999))

Uniformly concave metric spaces are Lipschitz-free rigid.

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Theorem (Mayer-Wolf (1981)/ Weaver (1999))

Uniformly concave metric spaces are Lipschitz-free rigid.

Theorem (Alexander, Fradelizi, García Lirola, Zvavitch (2020))

Let M be a finite metric space and let G be its *canonical graph* (i.e. an edge-weighted graph encoding the distances of M with minimal number of edges). If G is 3-connected, then M is Lipschitz-free rigid.
In general, every isometry of $\mathcal{F}(M)$ is induced by a cycle-preserving bijection of edges of G.

Let $G = (V, E)$ be a connected graph (of any cardinality). A *graph metric* on G is a metric defined on V where for $x, y \in V$

$$d(x,y) := \min\{n\colon \exists e_1,\ldots,e_n \in E\big(e_1\ldots e_n \text{ form an edge path between } x \text{ and } y\big)\}.$$

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Fact (Whitney ??)

A graph is 2-connected if and only if every two vertices lie on a common cycle.

Graph metric spaces

Let $G = (V, E)$ be a graph and let $E' \subseteq E$ be a subset of edges. Denote by $V(E')$ the vertices incident with E' and by $G(E') := (V(E'), E')$ the subgraph of G induced by E'.

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Define an equivalence relation $\sim$ on E by setting $e \sim f$, for $e, f \in E$ if e and f lie on a common cycle.
By Whitney's theorem, for every $e \in E$ and its equivalence class $[e]_\sim$ we have that $G([e]_\sim)$ is 2-connected.

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Proposition

Let $G = (V, E)$ be a graph. Let $(e_\alpha)_{\alpha \in I} \subseteq E$ be a selection of representatives of equivalence classes of $\sim$. Then

$$\mathcal{F}(G) \equiv \Big(\bigoplus_{\alpha \in I} \mathcal{F}(G([e_\alpha]_\sim)) \Big)_{\ell_1}.$$

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Define an equivalence relation on I by declaring $\alpha, \beta \in I$ to be equivalent if $\mathcal{F}(G([e_\alpha]_\sim)) \equiv \mathcal{F}(G([e_\beta]_\sim))$. Let $(I_n)_{n \in J}$ be an enumeration of the equivalence classes. For each $n \in J$

- let S_n be the permutation group of I_n;

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Proposition

The linear isometry group of $\mathcal{F}(G)$ is equal to

$$\prod_{n \in J} G_n \wr S_n \Big(= \big(\prod_{\alpha \in I_n} G_n \big) \rtimes S_n \Big).$$

Takeaway message

When determining the isometries of Lipschitz-free spaces over graphs it is enough to restrict to graphs that are 2-connected.

Graph metric spaces

Theorem

Let G be a 3-connected graph. Then G is Lipschitz-free rigid.

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Corollary/Remark

Notice that since $\mathcal{F}(\mathbb{Z}) \equiv \ell_1$, $\mathbb{Z}$, which can be viewed as a graph metric space, is very far from being Lipschitz-free rigid (its isometry group is a quite basic wreath product described by the Banach-Lamperti theorem or the proposition from the previous slide).
However, for $d \geq 2$ we have that $\mathbb{Z}^d$ as a graph, equivalently $\mathbb{Z}^d$ with the ℓ_1-metric, is Lipschitz-free rigid.

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Remark

We have a complete (combinatorial) description of linear isometries (and linear isometry groups) of Lipschitz-free spaces over 2-connected graphs.

2-connected graphs

Question

Are there graphs that are Lipschitz-free rigid but not 3-connected?

2-connected graphs

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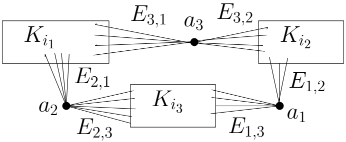

The graph consists of three disjoint complete graphs K_{i_1}, K_{i_2} and K_{i_3} together with three more vertices a_1, a_2 and a_3, where each a_j, $j \in \{1,2,3\}$ is connected by edges $E_{j,k}$ with vertices from K_{i_k}, $k \in \{1,2,3\}$, where $E_{j,j} = \emptyset$ and $|E_{j,k}| \geq 3$ are distinct natural numbers. If $\min\{i_1, i_2, i_3\} > 2 + \max\{|E_{(j,k)}| : j,k \in \{1,2,3\}, j \neq k\}$, then the graph is Lipschitz-free rigid and not 3-connected.

Part II

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Observe that if X is a Banach space, $\phi : X \to X$ a linear isometry, and $x \in X$ a preserved extreme point, then $\phi(x)$ is a preserved extreme point as well.

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Fact (Weaver)

A preserved extreme point of a Lipschitz-free space must be a (normalized) elementary molecule.

Fact (Aliaga, Guirao)

Let M be a metric space and $x \neq y \in M$. Then $n_{x,y}$ is preserved extreme if and only if the metric segment $[x, y]$ is 'uniformly trivial'.

Graph-theoretic notions

Let M be a metric space.
By $E_{ext}(M)$ we shall denote the set

$$\{(x, y) \in M^2 \colon n_{x,y} \text{ is a preserved extreme point in } B_{\mathcal{F}(M)}\}.$$

By $V_{ext}(M)$ we denote the set $\{x \in M \colon \exists y \in M \left((x, y) \in E_{ext}\right)\}$ and by $G_{ext}(M) := (V_{ext}(M), E_{ext}(M))$ the corresponding directed graph.

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For an edge e in any directed graph we denote by $s(e)$, resp. $r(e)$, the source, resp. the range of e, i.e. $e = (s(e), r(e))$.
For any elements x, y in a directed graph (V, E) an *edge path* from x to y is a sequence $(e_i)_{i=1}^n \subseteq E$ such that $s(e_1) = x$, $r(e_n) = y$, and $r(e_i) = s(e_{i+1})$ for $1 \leq i < n$.

Let M be a metric space. Say that $E_{ext}(M)$ is *admissible* if $V_{ext}(M)$ is dense in M and $G_{ext}(M)$ is connected.

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Theorem

Let M be a metric space such that $E_{ext}(M)$ is admissible and $G_{ext}(M)$ is 3-connected. Then M is Lipschitz-free rigid.

Definition

Call a metric space M a *Prague space* if $E_{ext}(M)$ is admissible and for every $x, y \in V_{ext}(M)$

$$d(x,y) = \inf\Big\{\sum_{i=1}^{n} d(e_i)\colon e_1,\ldots,e_n \text{ is } E\text{-path from } x \text{ to } y\Big\}.$$

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Proposition

Let M be a Prague space. Then there is a one-to-one bijection between surjective linear isometries of $\mathcal{F}(M)$ and symmetric bijections σ of $E_{ext}(M)$ preserving directed simple cycles and such that $\frac{d(e)}{d(\sigma(e))}$ is constant on each simple directed cycle.

Examples

Proposition

If M is a Prague space and $G_{ext}(M)$ is not 2-connected, then M is not Lipschitz-free rigid.

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Let $M' := [0,1] \subseteq \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ and set $M := M' \cup \{(0,-1)\}$.
Then $G_{ext}(M)$ is not 2-connected, however M is Lipschitz-free rigid.
Notice that M is not a Prague space.

Preservation results

Let M and N be metric space. Let $p \in [1, \infty)$ and denote by $M \oplus_p N$ the metric space $M \times N$ with the metric

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Theorem

Let M and N be metric spaces such that $|M| \geq 3$ and $E_{ext}(N)$ is admissible and let $p \in (1, \infty)$. Then $E_{ext}(M \oplus_p N)$ is admissible and $G_{ext}(M \oplus_p N)$ is 3-connected. In particular, $M \oplus_p N$ is Lipschitz-free rigid.

Corollary

Let N be a metric space containing three points that do not lie on one segment. Let M be any metric space. Then there exists a metric d on $M \coprod N$ such that $(M \coprod N, d)$ is Lipschitz-free rigid. In particular, every metric space can be isometrically embedded into a Lipschitz-free rigid metric space that has only three more elements.

Problems

Question 1

Does there exist a Lipschitz-free rigid metric space M such that $\bigcup E_{ext}(M)$ is not dense in M?

A very interesting answer to the previous question would be answering positively the next question. However, even a negative answer to the next question would be interesting.

Question 2

Is $\mathbb{R}^d$, for $d \geq 2$, Lipschitz-free rigid? Either if it is equipped with the euclidean or with the Manhattan distance.

Question 3

Does every metric space isometrically embed into a Lipschitz-free rigid space that contains only one additional point?

Examples of Lipschitz-free rigid Prague spaces- Carnot groups

A *Carnot group* is a simply connected nilpotent Lie group whose (real) Lie algebra ($\mathbb{R}^n$ as a vector space) admits a decomposition $V_1 \oplus \ldots V_k$ such that $[V_1, V_l] = V_{l+1}$, for $l \in \mathbb{N}$, where $V_l = \{0\}$ for $l > k$.

This implies that a Carnot group G is a topological group $(\mathbb{R}^n, *)$, where the group operation $*$ is defined by a polynomial. Moreover, we can decompose the Carnot group $(\mathbb{R}^n, *)$ as $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \ldots \oplus \mathbb{R}^{n_r}$ with $n_1 + \cdots + n_r = n$ and find a family of Carnot group isomorphisms $\{\delta_\lambda\}_{\lambda>0}$ (called dilations) such that:

$$\delta_\lambda(x^{(1)}, \ldots, x^{(r)}) = (\lambda x^{(1)}, \ldots, \lambda^r x^{(r)}),$$

where $x^{(i)} \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, r$.

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where $x^{(i)} \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, r$.

The elements $\{(z, 0, \ldots, 0) \colon z \in \mathbb{R}^{n_1}\}$ are called *horizontal*.

Carnot groups example

The elements $\{(z, 0, \ldots, 0) \colon z \in \mathbb{R}^{n_1}\}$ are called horizontal and subsets (or more precisely one-parameter subgroups) of the form

$$\{(sz, 0, \ldots, 0) \colon s \in \mathbb{R}\},$$

where $z \in \mathbb{R}^{n_1}$, are called *horizontal lines*.

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A *homogeneous norm* on G is a function $N : G \to [0, \infty)$ satisfying

(i) $N(g) = 0 \iff g = (0, \dots, 0)$,
(ii) $N(g^{-1}) = N(g)$, for all $g \in G$,
(iii) $N(g * g') \leq N(g) + N(g')$, for all $g, g' \in G$,
(iv) $N(\delta_\lambda(g)) = \lambda N(g)$ for $\lambda > 0$ and $g \in G$.

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$$\{(sz, 0, \ldots, 0)\colon s \in \mathbb{R}\},$$

where $z \in \mathbb{R}^{n_1}$, are called *horizontal lines*.
A *homogeneous norm* on G is a function $N : G \to [0, \infty)$ satisfying

(i) $N(g) = 0 \iff g = (0, \ldots, 0)$,
(ii) $N(g^{-1}) = N(g)$, for all $g \in G$,
(iii) $N(g * g') \leq N(g) + N(g')$, for all $g, g' \in G$,
(iv) $N(\delta_\lambda(g)) = \lambda N(g)$ for $\lambda > 0$ and $g \in G$.

As in Banach spaces, homogeneous norms define a (left-invariant) distance on G compatible with its topology.

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As in Banach spaces, homogeneous norms define a (left-invariant) distance on G compatible with its topology.
N is *horizontally strictly convex* if whenever $g, h \in G$ are such that $N(gh) = N(g) + N(h)$, then there is a horizontal line $L \subseteq G$ such that $g, h \in L$.

Carnot groups example

Theorem

Let G be a non-abelian Carnot group equipped with a horizontally strictly convex norm. Then G is a Lipschitz-free rigid Prague space.

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Remark 1

Horizontally strictly convex norms have been extensively studied especially on Heisenberg group(s) (e.g. the Heisenberg-Koranyi norm or the Lee-Naor norm). However, there is a horizontally strictly convex norm on every Carnot group (e.g. the so-called Hebisch-Sikora norm).

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Remark 2

Unless G is abelian, the horizontally strictly convex norm is not geodesic although it contains large geodesic subspaces (the restriction of the norm on the horizontal elements is often Euclidean).

Isometry groups as isometry invariants

Remark

It is possible to distinguish $\mathcal{F}(\mathbb{Z}^n)$ and $\mathcal{F}(\mathbb{Z}^m)$, for $n \neq m \geq 2$, isometrically using the linear isometry groups as invariants. Indeed, both $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are Lipschitz-free rigid and it is easy to verify that the isometry groups of $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are different.
As R. Aliaga pointed out, showing $\mathcal{F}(\mathbb{Z}^n) \not\equiv \mathcal{F}(\mathbb{Z}^m)$ can be done much simpler just by counting the preserved extreme points. However, the next example is more involved.

Isometry groups as isometry invariants

Recall that the Heisenberg group $\mathbb{H}^n$ is the Carnot group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, *)$, where for $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we have

$$(x, y, t) * (x', y', t') := (x + x', y + y', t + t' + 2\sum_{i=1}^{n}(x'_i y_i - x_i y'_i))$$

and Carnot-dilations $(\delta_\lambda)_{\lambda>0}$ are given by $\delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$. An examples of horizontally strictly convex homogeneous norm on $\mathbb{H}^n$ is e.g. the Heisenberg-Korányi norm $\|\cdot\|_H$ given by $\|(x, y, t)\|_H := \left(\|(x, y)\|_E^4 + t^2\right)^{\frac{1}{4}}$ for $(x, y, t) \in \mathbb{H}^n$.

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Fact

For $n \neq m$, $\mathcal{F}(\mathbb{H}^n) \not\equiv \mathcal{F}(\mathbb{H}^m)$.