A new presentation of Pełczyński's universal space

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(included in **The Zoo of combinatorial Banach spaces**, a joint work with P. Borodulin-Nadzieja, S. Jachimek, and A. Pelczar-Barwacz)

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Pełczyński constructed a normalized unconditional basis (x_n) such that every such basis (y_k) in a Banach space Y is (permutatively) equivalent to a subbasis (x_{n_k}) of (x_n). In other words, this basis itself is universal for all normalized unconditional bases.

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(b) (x_n) is not universal, (e.g.) no subbasis (x_{n_k}) is equivalent to the canonical basis of ℓ_2.

(b') If (x_{n_k}) is a subbasis, then $[x_{n_k}]$ contains either c_0 or ℓ_1.

Combinatorial Banach spaces

Given a hereditary $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty} = \{$finite subsets of $\mathbb{N}\}$ covering $\mathbb{N}$ let

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$$\begin{aligned} \|x\|_{\mathcal{F}} &= \sup\left\{\sum_{k\in F}|x(k)| : F \in \mathcal{F}\right\} \ (x \in \mathbb{R}^{\mathbb{N}}), \\ X_{\mathcal{F}} &= \text{the completion of } \left(c_{00}, \|\bullet\|_{\mathcal{F}}\right) \\ &= \left\{x \in \mathbb{R}^{\mathbb{N}} : \left\|P_{[n,\infty)}(x)\right\|_{\mathcal{F}} \xrightarrow{n\to\infty} 0\right\}. \end{aligned}$$

The canonical basis (e_n) of c_{00} is a 1-unconditional basis in $X_{\mathcal{F}}$, and if (e_{n_k}) is a subbasis, then $[e_{n_k}]$ contains either c_0 or ℓ_1.

Examples

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$\mathcal{C} = \{C \subseteq 2^{<\mathbb{N}} : C \text{ is a finite chain}\} \rightsquigarrow X_{\mathcal{C}} =$ [dual functionals to the basis of $X_{\mathcal{A}}$], it contains copies of all separable Banach spaces with unconditional basis.

Pełczyński's space of the form $X_{\mathcal{P}}$

We would like to find a hereditary $\mathcal{P} \subseteq [\mathbb{N}]^{<\infty}$ covering $\mathbb{N}$ such that $X_{\mathcal{P}}$ contains complemented copies of every separable Banach space with unconditional basis.

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(i) Complemented copy in some $X_{\mathcal{F}}$: For every separable Y with unconditional basis there is an $\mathcal{F}$ such that $X_{\mathcal{F}}$ contains a complemented copy of Y.

(ii) A universal hereditary hypergraph: There is a hereditary $\mathcal{P} \subseteq [\mathbb{N}]^{<\infty}$ covering $\mathbb{N}$ such that every such $\mathcal{F}$ is isomorphic to $\mathcal{P} \restriction I = \{P \in \mathcal{P} : P \subseteq I\}$ for some $I \subseteq \mathbb{N}$ (and hence $[e_n : n \in I]$ is a complemented isometric copy of $X_{\mathcal{F}}$ in $X_{\mathcal{P}}$).

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- $\left\| \sum_n \alpha(n) x_n \right\|_{\mathcal{F}} \leq \|\alpha\|_Y \leq \frac{1}{1-\varepsilon} \left\| \sum_n \alpha(n) x_n \right\|_{\mathcal{F}}$ for every $\alpha \in \mathbb{R}^{\mathbb{N}}$,

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  hence (x_n) and (y_n) are equivalent.
- $T : X_{\mathcal{F}} \to [x_n]$, $T(x)(k) = \frac{1}{|I_n|} \sum_{i \in I_n} x(i)$ for $k \in I_n$, is a projection.

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We can construct a $\mathcal{P}$ satisfying the following extension property (which implies that $\mathcal{P}$ is a universal hereditary hypergraph):

IF $D \subseteq \mathbb{N}$ is finite, $\mathcal{Q} \subseteq \mathcal{P}(D \cup \{\infty\})$ is hereditary and covers $D \cup \{\infty\}$, and $\mathcal{Q} \restriction D = \mathcal{P} \restriction D$,

THEN there is an $n \notin D$ such that $e : D \cup \{\infty\} \to D \cup \{n\}$, identity on D and $e(\infty) = n$, is an isomorphism between $\mathcal{Q}$ and $\mathcal{P} \restriction (D \cup \{n\})$.

Thank you!