# On Mazur rotations problem and its topological and multidimensional aspects

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# Main cast

$X$=Banach space, infinite dimensional unless specified otherwise, $S_X$=unit sphere of $X$, $B_X$=unit ball of $X$

$H$=the Hilbert space

$GL(X)$ = group of linear automorphisms on $X$

$\mathrm{Isom}(X)$ = group of (surjective) linear isometries on $X$

$\mathrm{Age}(X)$ = the set of finite-dimensional subspaces of $X$
$\mathrm{Age}_n(X)$ = the set of $n$-dimensional subspaces of $X$

$\mathcal{L}(Y, X)$ = the space of bounded operators from $Y$ into $X$
$\mathrm{Emb}(Y, X)$ = the set of isometric linear embeddings of $Y$ into $X$

# Topological groups

## Definition

*A topological group is a group with a topology such that*

- $(g, h) \mapsto g.h$ *is continuous from* $G \times G$ *to* $G$
- $g \mapsto g^{-1}$ *is continuous from* $G$ *to* $G$

## Example

- *Every group* $G$ *is topological with the discrete topology.*
- *Every subgroup of a topological group is topological with the induced topology.*

# Topologies on $GL(X)$

Several topologies can be relevant on $GL(X)$:

- the norm topology $\|T\| = \sup_{x \in S_X} \|Tx\|$.
- the $\mathrm{SOT}$ (strong operator topology) i.e. pointwise convergence on $X$, i.e.,

$$T_\alpha \to T \Leftrightarrow T_\alpha(x) \to T(x), \forall x \in X.$$

- (maybe not for this minicourse) the $\mathrm{WOT}$ (weak operator topology) i.e. weak ptwise cv on $X$, but also $\mathrm{SOT}^*, \mathrm{SOT}_*, \ldots$

# Topologies on $GL(X)$

**Fact**

$(GL(X), \|.\|)$ *is a topological group*

If $T_n \to T$ then $\|T_n^{-1}\| \to 1$ and $T_n^{-1} - T^{-1} = T_n^{-1}(T - T_n)T^{-1}$

$(GL(X), SOT)$ is not a topological group in general, but things get better for bounded subgroups (in particular $\mathrm{Isom}(X)$).

**Fact**

$\mathrm{Isom}(X)$ *is a topological group for SOT.*

(An easy exercise)

# Separability

$(\mathrm{Isom}(X), \|.\|)$ is also a topological group. But $(\mathrm{Isom}(X), \|.\|)$ is not separable in general, even for $X$ separable.

## Example

*If $X = \ell_p$, $1 \leq p < +\infty$, and $\alpha = (\alpha_n)_n \in \{-1, 1\}^{\mathbb{N}}$, then*

$$T_\alpha(\lambda_1, \lambda_2, \ldots) = (\alpha_1\lambda_1, \alpha_2\lambda_2, \ldots)$$

*defines an uncountable family of isometries such that $\|T_\alpha - T_\beta\| = 2$ if $\alpha \neq \beta$.*

## Fact

*If $X$ is separable then $(\mathrm{Isom}(X), SOT)$ is separable.*

Indeed if $D$ is a dense countable family in $X$, then $(\mathrm{Isom}(X), SOT)$ is homeomorphic (by $T \mapsto T_{|D}$) to $(\mathrm{Emb}_d(D, X), SOT) \subset X^D$, where $\mathrm{Emb}_d(D, X)$ is the space of linear isometric dense embeddings of $D$ into $X$.

# Polish groups

## Definition

- *A Polish space is a separable, completely metrizable, topological space.*
- *A Polish group is a topological group whose topology is Polish.*

## Example

$(\mathbb{R}, +)$ *is a Polish group,* $(]0, +\infty[, .)$ *also (via the exponential map).*

## Fact

*If $X$ is separable, then* $(\mathrm{Isom}(X), SOT)$ *is a Polish group.*

## Proof.

Next slide

# Proof that $\mathrm{Isom}(X)$ is Polish when $X$ is separable

- It is a separable topological group, homeomorphic to $\mathrm{Emb}_d(D, X) \subset X^D$, with $D = \{d_n, n \in \mathbb{N}\}$ dense, which is metrizable through

$$d(T, U) = \sum_n \min\{2^{-n}, d(Td_n, Ud_n)\}$$

- but this metric is not necessarily complete (a sequence of surjective isometries could converge pointwise to a **non surjective** isometric embedding).
- We note that $d(T^{-1}, U^{-1})$ is a compatible metric (since $T \mapsto T^{-1}$ is continuous) and that

$$D(T, U) = d(T, U) + d(T^{-1}, U^{-1})$$

is a complete compatible metric (clue: if $T_n \to^{SOT} T$ and $T_n^{-1} \to^{SOT} U$ in $B_{\mathcal{L}(X)}$ then $T$ is surjective and $T^{-1} = U$).

# The Polish group $\mathrm{Isom}(X)$

Summing up:

$\mathrm{Isom}(X)$ will always be equipped with the Strong Operator Topology SOT.

This topology turns it into a topological group, and even a Polish group when $X$ is separable.

Regarding $\mathrm{Emb}(F, X)$, for $F$ finite dimensional, we shall usually equip $\mathrm{Emb}(F, X)$ with the distance induced by the norm on $\mathcal{L}(F, X)$. But note that here SOT and the norm topology are equivalent.

# Orbits

If $G$ is a group of isometries on $X$, then we denote $\mathrm{Orb}_G(x)$ the orbit of the point $x$ of $X$ under the action of the group $G$, i.e.

$$\mathrm{Orb}_G(x) = Gx := \{gx, g \in G\}.$$

When $G = \mathrm{Isom}(X, \|.\|)$, then we denote the orbit of $x$ under $G$ simply by $\mathrm{Orb}(x)$, i.e.

$$\mathrm{Orb}(x) = \{gx, g \in \mathrm{Isom}(X, \|.\|)\}.$$

# Classical isometry groups

1. If $H$=Hilbert, then $\mathrm{Isom}(H)$ is the unitary group $\mathcal{U}(H)$. It acts transitively on $S_H$, meaning there is a single orbit for the action $\mathrm{Isom}(H) \curvearrowright S_H$, i.e. $\mathrm{Orb}(x) = S_H$ for all $x \in S_H$.

2. For $1 \leq p < +\infty$, $p \neq 2$, every isometry on $L_p = L_p(0,1)$ is of the form
$$T(f)(.) = \varepsilon(.)h(.)f(\phi(.)),$$
$\phi$ is a measurable transformation of $[0,1]$ onto itself, $h = (d(\lambda \circ \phi)/d\lambda)^{\frac{1}{p}}$, $\lambda$ the Lebesgue measure, and $\varepsilon$ unimodular (Banach-Lamperti 1932-1958). Therefore:

3. $\mathrm{Isom}(L_p)$ acts almost transitively on $S_{L_p}$, meaning that the action $\mathrm{Isom}(L_p) \curvearrowright S_{L_p}$ admits dense orbits, $\overline{\mathrm{Orb}(x)} = S_{L_p}$ for all $x \in S_{L_p}$. Proof of 3:

# Classical isometry groups: $\mathrm{Isom}(L_p)$ has dense orbits

## Proof

*Let $h > 0$ belong to $S_{L_p}$. Let $\phi(x) = \int_0^x h(t)^p dt$ and*

$$T_h(f)(.) = h(.)f(\phi(.))$$

*Then $T_h$ is a linear isometry sending 1 to $h$. So $h \in \mathrm{Orb}(1) := \{T(1) : T \in \mathrm{Isom}(L_p)\}$. Using change of signs,*

$$S_{full} := \{h \in S_{L_p} : h(t) \neq 0 \text{ a.e.}\} \subseteq \mathrm{Orb}(1)$$

*On the other hand it is clear that $\overline{S_{full}} = S_{L_p}$. So the action of $\mathrm{Isom}(L_p)$ on the sphere has dense orbits.*

On the other hand $\mathrm{Isom}(L_p)$ does not act transitively on the sphere since (exercise) $\mathrm{Orb}(1) = S_{full}$; actually there are exactly 2 orbits.

# Classical isometry groups

4 Every isometry on $c_0$ and $\ell_p$, $p \neq 2$ acts as a ”signed permutation”, i.e. a combination of signs and permutation of the coordinates on the canonical basis.

5 By the Banach-Stone theorem (1932), every isometry of $C(K)$ is of the form

$$T(f)(.) = h(.)f(\phi(.)),$$

where $h$ is continuous unimodular on $K$ and $\phi$ a homeomorphism of $K$.

6 It follows that $\mathrm{Isom}(\ell_p)$ (resp. $\mathrm{Isom}(c_0)$, resp. $\mathrm{Isom}(C(K))$) do not act almost transitively on the sphere. This somehow tells us that these spaces are too rigid, or their isometry group is not “big enough”.

# Examples of almost transitive spaces

## Definition

*A space $X$ is almost transitive* $\Leftrightarrow \exists x \in S_X$ *such that* $\mathrm{Orb}(x)$ *is dense in* $S_X \Leftrightarrow \forall x \in S_X$, $\mathrm{Orb}(x)$ *is dense in* $S_X$

## Examples

- $L_p, 1 \leq p < \infty$
- *Gurarij space G (1966)*
- $Z_X$*: any separable Banach space $X$ is complemented in some separable almost transitive space $Z_X$ (Lusky 79)*
- *and therefore some example $Z_X$ without AP*
- $L_p(X)$ *whenever $X$ is almost transitive (Greim Jamison Kaminska 94)*

# Mazur rotations problem

We have the following problem appearing in Banach's book "Théorie des opérations linéaires", 1932.

## Problem

*If $G = \mathrm{Isom(X)}$ acts transitively on $S_X$, must $X$ be isometric? isomorphic? to a Hilbert space.*

(a) if $\dim X < +\infty$: YES to both
(b) if $\dim X = +\infty$ and is separable: ???
(c) if $\dim X = +\infty$ and is non-separable: NO to both

## Proof

*Proof of (a)*

# Finite dimensional transitive spaces are euclidean

## Proof

- *Let* $\dim X < +\infty$. *Let* $< .,. >$ *be any inner product on* $X$ *such that (1)* $\|x_0\| = \sqrt{< x_0, x_0 >}$ *for some given* $x_0 \neq 0$.
- *Since* $G = \mathrm{Isom}(X, \|.\|)$ *is a compact group, consider the invariant means* $\mu$ *associated to the Haar measure on it, i.e. a positive linear functional such that whenever* $f : G \to \mathbb{R}$ *is continuous, then* $\mu(f) = \mu(g.f)$, *where* $g.f$ *is defined by* $g.f(h) = f(g^{-1}h)$.
- *Define*

$$[x, y] = \mu(g \mapsto < gx, gy >) = \int_{g \in \mathrm{Isom}(X, \|.\|)} < gx, gy > d\mu(g),$$

*This is a new inner product, inducing an equivalent Hilbert norm for which all g's in* $\mathrm{Isom}(X), \|.\|$ *are again isometries.*

# Finite dimensional transitive spaces are euclidean

- So from (1),

$$\|gx_0\| = \sqrt{< gx_0, gx_0 >} \forall g \in \mathrm{Isom}(X, \|.\|),$$

  i,e.,

$$\|x\| = \sqrt{< x, x >}, \forall x \in Orb(x_0).$$

- If $X$ was transitive, then this holds for all $x \in X$. So $X, \|.\|$ is Hilbertian.
- Actually, almost transitivity is obviously enough.

## Problem

*If $G = \mathrm{Isom(X)}$ acts transitively on $S_X$, must $X$ be isometric? isomorphic? to a Hilbert space.*

(a) if $\dim X < +\infty$: YES to both
(b) if $\dim X = +\infty$ and is separable: ???
(c) if $\dim X = +\infty$ and is non-separable: NO to both

Let us give a proof of (c).

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and $\lim_{\mathcal{U}}$ associated, i.e. this is a Hahn-Banach extension to $\ell_\infty$ of the functional $\phi$ defined by $\phi(x) = \lim_n x_n$ for $x \in c$.

### Definition (Ultrapower $X_{\mathcal{U}}$)

*If $X$ is a Banach space, let* $\mathrm{Null}_{\mathcal{U}}(X) = \{(x_n)_n \in \ell_\infty(X) : \lim_{\mathcal{U}} x_n\} = 0$*, and let*

$$X_{\mathcal{U}} := \ell_\infty(X)/\mathrm{Null}_{\mathcal{U}}(X),$$

*This is a Banach space under the norm* $\|(x_n)_n\| := \lim_{\mathcal{U}} \|x_n\|$.

### Observation

*If $X$ is almost transitive then $X_{\mathcal{U}}$ is transitive, under the action of the subgroup $\mathrm{Isom}(X)_{\mathcal{U}}$ of isometries $T$ of the form*

$$T((x_n)_{n\in\mathbb{N}}) = (T_n(x_n))_{n\in\mathbb{N}}, \text{where}\, T_n \in \mathrm{Isom}(X).$$

# Ultrapowers

As an immediate corollary we deduce:

### Corollary

*The space $(L_p(0,1))_{\mathcal{U}}$ is a non-separable transitive space, non-isomorphic to a Hilbert space if $p \neq 2$.*

Note that in these lines Cabello-Sanchez (1998) studies $\Pi_{n\in\mathbb{N}} L_{p_n}(0,1)$ for $p_n \to +\infty$ and obtains a transitive M-space.

In any case, this gives a negative answer to (c), i.e. the nonseparable Mazur rotations problem.

# Isomorphic and isometric version of Mazur's problem

We can split Mazur rotations problem in two open problems.

### Problem (Isomorphic Mazur problem)

*If $X, \|.\|$ is separable transitive, then must it be isomorphic to Hilbert space $H$?*

### Problem (Isometric Mazur problem)

*If $\|.\|$ is an equivalent transitive norm on Hilbert space $H$, must it be an Hilbertian norm (i.e. be induced by an inner product)?*

# The isometric Mazur problem is a question in topological group theory

The proof of Mazur problem in finite dimension using the Haar measure on $G = \mathrm{Isom}(X)$ will work when $\dim X = \infty$ a soon as

- $X$ is isomorphic to a Hilbert space (with inner product $< .,. >$)
- we can give a meaning to the expression $\mu(f) = \int_{g \in G} < gx, gy > d\mu(g)$ as an invariant means.

## Definition

- *A map $f : G \to \mathbb{R}$ is* left uniformly continuous *if* $\forall \varepsilon > 0, \exists V$ *neighborhood of $e_G$ such that*

$$g^{-1}h \in V \Rightarrow |f(g) - f(h)| \leq \varepsilon$$

- *A topological group $G$ is* topologically amenable *if there exists an invariant means $\mu$ defined on the set of uniformly continuous maps from $G$ to $\mathbb{R}$.*

# The isometric Mazur problem is a question in topological group theory

We deduce

## Proposition

*Assume $X$ is an almost transitive equivalent renorming of the Hilbert space, for which $\mathrm{Isom}(X)$ is topologically amenable. Then $X$ is isometric to a Hilbert space.*

## PROOF.

Just note that $f_{x,y} : g \mapsto < gx, gy >$ is uniformly continuous from $(\mathrm{Isom}(X), \mathrm{SOT})$ into $\mathbb{R}$: indeed, define the SOT-neighborhood $U_{x,y,\varepsilon}$ of $Id_X$ by $g \in U_{x,y,\varepsilon} \Leftrightarrow \|gx - x\| < \varepsilon, \|gy - y\| < \varepsilon$. Then

$$g^{-1}h \in U_{x,y,\varepsilon} \Rightarrow \|gx - hx\| \leq \varepsilon, \|gy - hy\| \leq \varepsilon \Rightarrow$$

$$|f_{x,y}(g) - f_{x,y}(h)| = |< gx, gy > - < hx, hy >| \leq K\varepsilon.$$

# Amenabilities of topological groups

The following formulation evidentiates the relation between notions of amenabilities. Let $G$ be a topological group. Then

## Definition

- *$G$ is amenable ("as a discrete group") if any affine action on a non-empty compact convex subset of a topological vector space has a fixed point*
- *$G$ is topologically amenable if any continuous affine action on a non-empty compact convex subset of a topological vector space has a fixed point*
- *$G$ is extremely amenable if any continuous action on a non-empty compact space has a fixed point*

For example, compact groups are topologically amenable, but not necessarily amenable.

# Isometric Mazur problem and topological amenability

Finally:

**Theorem**

*Let $X$ be an (almost) transitive renorming of the Hilbert space. Then the following are equivalent:*

1. *$X$ is isometric to a Hilbert space*
2. $\mathrm{Isom}(X)$ *is topologically amenable*
3. $\mathrm{Isom}(X)$ *is extremely amenable*

PROOF.

3. implies 2. is obvious, and 2. implies 1. was just observed.
1. implies 3., i.e. that $\mathcal{U}(H)$ is SOT - extremely amenable, is due to Gromov - Milman 83, using concentration of measure. □

# Mazur problem and multidimensionality: a first glimpse

$\mathrm{Age}(X) \equiv$ the set of finite-dimensional subspaces of a Banach space $X$,
$\mathrm{Age}_n(X) \equiv$ the set of $n$-dimensional subspaces of a Banach space $X$,
and, for $F \in \mathrm{Age}(X)$, $\mathrm{Emb}(F, X) \equiv$ the set of isometric (linear) embeddings of $F$ into $X$.

## Definition

*An infinite dimensional Banach space is* ultrahomogeneous *(or* ultratransitive*) if for any $F \in \mathrm{Age}(X)$, any $i, j \in \mathrm{Emb}(F, X)$, there exists $T \in \mathrm{Isom}(X)$ such that $T \circ i = j$.*

Note that every infinite dimensional Hilbert space is ultrahomogeneous: in other words not only all $n$-dimensional subspaces have the same shape, but they are also in the same position inside it; we could say Hilbert space is $n$-dimensionally isotropic.

# The multidimensional Mazur problem

Problem ("Multidimensional Mazur problem", still open)

*Show that every separable ultrahomogeneous Banach space is Hilbertian.*

- it is clear that the answer is positive in the finite dimensional case
- What about the non-separable case?

Question

*What about the non-separable transitive spaces* $(L_p)_{\mathcal{U}}$*? Are they ultrahomogeneous?*

# Non-separable ultrahomogeneous spaces

One could expect the non-separable transitive spaces $(L_p)_{\mathcal{U}}$ to be ultrahomogeneous. However one is taken by surprise as

### Theorem

- *the space* $(L_p)_U$ *is ultrahomogeneous if* $p \neq 4, 6, 8, \ldots$ *(F. Lopez-Abad Mbombo Todorcevic 20)*
- *if* $p = 4, 6, 8, \ldots$ *then it is transitive but not ultrahomogeneous (follows from Randrianantoanina 99)*

These are non-separable (reflexive) counterexamples to Multidimensional Mazur problem, and so this question is really a separable problem. (note that a (non-reflexive) non-separable ultrahomogeneous example had been obtained in 2016 by Aviles, Cabello, Castillo, Gonzalez, and Moreno, namely $G_{\mathcal{U}}$).

We shall come back later to the Lp situation.

# On renormings of classical spaces

Recall that for $p \neq 2$, $L_p$ is not transitive, and $\ell_p$ not almost transitive. Furthermore

**Theorem (Dilworth - Randrianantoanina, 2014)**

*Let $1 < p < +\infty, p \neq 2$. Then*
*$\ell_p$ does not admit an equivalent almost transitive norm.*

(a first very exotic superreflexive example with no almost transitive renorming had been obtained by F. - Rosendal, 2013)

**Question**

*Let $1 \leq p < +\infty, p \neq 2$. Show that the space $L_p(0,1)$ does not admit an equivalent transitive norm.*

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