# On Mazur rotations problem and its topological and multidimensional aspects Part 2

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# Recall from last time, when $\dim X < +\infty$

- Let $G = \mathrm{Isom}(X, \|.\|)$. The formula

$$[x, y] = \mu(g \mapsto < gx, gy >) = \int_G < gx, gy > d\mu(g),$$

  using the Haar measure, is an equivalent inner product on $X$ which is $G$-invariant ($[gx, gy] = [x, y]$).
- so, if $x_0 \in S_X$ and $\lambda$ is such that $\|x_0\| = \lambda\sqrt{[x_0, x_0]}$, then

$$\|x\| = \lambda\sqrt{[x, x]}$$

  whenever $x \in \mathrm{Orb}_G(x_0)$.
- if $X$ was (almost) transitive, then this holds for all $x \in X$, so $\|.\|$ is a Hilbert norm.

So Mazur rotations problem has a positive answer if $\dim X < \infty$.

# The isometric Mazur problem is a question in topological group theory

The previous proof in finite dimension using the Haar measure on $G = \mathrm{Isom}(X)$ will work when $\dim X = \infty$ a soon as

- $X$ is isomorphic to a Hilbert space (with inner product $< .,. >$)
- we can give a meaning to the expression $\mu(f) = \int_{g \in G} < gx, gy > d\mu(g)$ as a $G$-invariant means.

## Definition

- *A map* $f : G \to \mathbb{R}$ *is* left uniformly continuous *if* $\forall \varepsilon > 0, \exists V$ *neighborhood of* $e_G$ *such that*

$$g^{-1}h \in V \Rightarrow |f(g) - f(h)| \leq \varepsilon$$

- *A topological group* $G$ *is* topologically amenable *if there exists an invariant means* $\mu$ *defined on the set of uniformly continuous maps from* $G$ *to* $\mathbb{R}$.

# The isometric Mazur problem is a question in topological group theory

We deduce

## Proposition

*Assume $X$ is an almost transitive equivalent renorming of the Hilbert space, for which $\mathrm{Isom}(X)$ is topologically amenable. Then $X$ is isometric to a Hilbert space.*

## Proof.

Just note that $f_{x,y} : g \mapsto < gx, gy >$ is uniformly continuous from $(\mathrm{Isom}(X), \mathrm{SOT})$ into $\mathbb{R}$: indeed, define the SOT-neighborhood $U_{x,y,\varepsilon}$ of $Id_X$ by $g \in U_{x,y,\varepsilon} \Leftrightarrow \|gx - x\| < \varepsilon, \|gy - y\| < \varepsilon$.
Then

$$g^{-1}h \in U_{x,y,\varepsilon} \Rightarrow \|gx - hx\| \leq \varepsilon, \|gy - hy\| \leq \varepsilon \Rightarrow$$

$$|f_{x,y}(g) - f_{x,y}(h)| = | < gx, gy > - < hx, hy > | \leq K\varepsilon. \qquad \square$$

# Amenabilities of topological groups

The following formulation evidentiates the relation between notions of amenabilities. Let $G$ be a topological group. Then

## Definition

- *$G$ is amenable ("as a discrete group") if any affine action on a non-empty compact convex subset of a topological vector space has a fixed point*
- *$G$ is topologically amenable if any continuous affine action on a non-empty compact convex subset of a topological vector space has a fixed point*
- *$G$ is extremely amenable if any continuous action on a non-empty compact space has a fixed point*

For example, compact groups are topologically amenable, but not necessarily amenable.

# Isometric Mazur problem and topological amenability

Finally:

**Proposition**

*Let $X$ be an (almost) transitive renorming of the Hilbert space. Then the following are equivalent:*

1. *$X$ is isometric to a Hilbert space*
2. $\mathrm{Isom}(X)$ *is topologically amenable*
3. $\mathrm{Isom}(X)$ *is extremely amenable*

PROOF.
3. implies 2. is obvious, and 2. implies 1. was just observed.
1. implies 3., i.e. that $\mathcal{U}(H)$ is SOT - extremely amenable, is due to Gromov - Milman 83, using concentration of measure. □

# Multidimensional Mazur problem: a first glimpse

$\mathrm{Age}(X) \equiv$ the set of finite-dimensional subspaces of a Banach space $X$,
and, for $F \in \mathrm{Age}(X)$, $\mathrm{Emb}(F, X) \equiv$ the set of isometric (linear) embeddings of $F$ into $X$.

## Definition

*An infinite dimensional Banach space is* ultrahomogeneous *(or* ultratransitive*) if for any $F \in \mathrm{Age}(X)$, any $i, j \in \mathrm{Emb}(F, X)$, there exists $T \in \mathrm{Isom}(X)$ such that $T \circ i = j$.*

Note that every infinite dimensional Hilbert space is ultrahomogeneous: in other words not only all $n$-dimensional subspaces have the same shape, but they are also in the same position inside it; we could say Hilbert space is $n$-dimensionally isotropic.

# The multidimensional Mazur problem

**Problem ("Multidimensional Mazur problem", still open)**

*Show that every separable ultrahomogeneous Banach space is Hilbertian.*

- it is clear that the answer is positive in the finite dimensional case
- What about the non-separable case?
  Precisely, what about the non-separable transitive spaces $(L_p)_{\mathcal{U}}$? Are they ultrahomogeneous?

One could expect theses spaces to be ultrahomogeneous. Or maybe not, for all values of $p$. However one is taken by surprise as ...

# Non-separable ultrahomogeneous spaces

## Theorem

- *the space* $(L_p)_U$ *is ultrahomogeneous if* $p \neq 4, 6, 8, \ldots$ *(F. Lopez-Abad Mbombo Todorcevic 20)*
- *if* $p = 4, 6, 8, \ldots$ *then it is transitive but not ultrahomogeneous (follows from Randrianantoanina 99)*

These are non-separable (reflexive) counterexamples to Multidimensional Mazur problem, and so this question is really a separable problem.
(Note that a (non-reflexive) non-separable ultrahomogeneous example had been obtained in 2016 by Aviles, Cabello, Castillo, Gonzalez, and Moreno, namely $\mathbb{G}_{\mathcal{U}}$).

We shall come back to this curious $L_p$ situation in the third talk.

# Digression: on renormings of classical spaces

Recall that for $p \neq 2$, $L_p$ is not transitive, and $\ell_p$ not almost transitive. Furthermore

**Theorem (Dilworth - Randrianantoanina, 2014)**

*Let $1 < p < +\infty, p \neq 2$. Then $\ell_p$ does not admit an equivalent almost transitive norm.*

(a first very exotic superreflexive example with no almost transitive renorming had been obtained by F. - Rosendal, 2013)

**Question**

*Let $1 \leq p < +\infty, p \neq 2$. Show that the space $L_p(0,1)$ does not admit an equivalent transitive norm.*

And now to a different perspective....

If $M$ is a metric space and $G$ acts isometrically on $M$, then for $m \in M$ define $[m] := \overline{\mathrm{Orb}}(m)$ and $M /\!/ G = (\{[m]\}, d)$ with the quotient metric $d([m].[n]) = \inf_{g \in G} d(gm, n)$.

### Fact

*The metric space $M /\!/ G$ is complete when $M$ is.*

### Definition

*Let $X$ be a Banach space. The isotropic quotient $\mathcal{I}_1(X)$ is $B_X /\!/ \mathrm{Isom}(X)$, equipped with the complete quotient metric $d(\overline{x}, \overline{y}) = inf_{g \in \mathrm{Isom}(X)} \|x - gy\|$.*

### Observation

*Note that $X$ is almost transitive $\Leftrightarrow S_X /\!/ \mathrm{Isom}(X) = \{*\}$, and in particular in this case $\mathcal{I}_1(X) \equiv [0, 1]$ via $x \mapsto \|x\|$.*

What else can happen for $\mathcal{I}_1(X)$?

# The case of $C(K)$

Recall:

**Theorem (Banach-Stone 32)**

*Every isometry of $C(K)$ is of the form*

$$T(f)(.) = h(.)f(\phi(.)),$$

*where $h$ is continuous unimodular on $K$ and $\phi$ a homeomorphism of $K$.*

Observe that $\mathrm{Orb}(1) = \{unimodular\ functions\}$, and so since $C(K)$ admits unimodular functions with values 1 and $-1$ unless $|K| = 1$ (Tietze-Urysohn), it is not almost transitive. i.e. the space $S_{C(K)} /\!/ \mathrm{Isom}(C(K))$ is not a singleton. But is it small or big?

We aim to describe this metric space, or equivalently, $\mathcal{I}_1(C(K))$.

# The case of $C(K)$

To avoid technicalities further on, we first consider the action of $G = \mathrm{Isom}_{lat}(C(K))$ :=the subgroup of isometries $T_\phi$ defined by $T_\phi(f) = f \circ \phi$.
Consider the map

$$\alpha : (C(K), \|.\|_\infty) \to (\mathcal{K}^*(\mathbb{R}), d_H)$$

defined by

$$\alpha(f) = f(K).$$

- It is 1-Lipschitz.
- It follows from the definition of $\mathrm{Isom}_{lat}(C(K))$ that if $g \in Orb_G(f)$, i.e. if there is $T \in \mathrm{Isom}_{lat}(C(K))$ such that $T(f) = g$, then $\alpha(f) = \alpha(g)$.
- by continuity, if $g \in [f] = \overline{Orb_G(f)}$, then $\alpha(f) = \alpha(g)$.

So the following is well-defined and 1-Lipschitz:

**Definition**

*Let*

$$\tilde{\alpha} : (C(K) \,/\!\!/\, \mathrm{Isom}_{lat}(C(K)), d) \to (\mathcal{K}^*(\mathbb{R}), d_H)$$

*be defined by*

$$\tilde{\alpha}([f]) := \alpha(f) = f(K).$$

Then

- It is 1-Lipschitz
- If $S$ denotes the subspace of simple functions in $C(K)$, then $\alpha(S) \subseteq \mathcal{F}^*(\mathbb{R}) := \{\textit{finite non empty subsets of } \mathbb{R}^+\}$.

Of course $\alpha$ can be far from surjective, for arbitrary $K$.

From now on we assume $K = 2^\omega$, the Cantor space.

# The case of $C(2^\omega)$

From now on we assume $K = 2^\omega$. We shall use the following facts.

- for all $n \in \mathbb{N}$, $2^\omega$ may be written as the union of $n$ clopen sets,
- All clopen subsets of $2^\omega$ are homeomorphic to $2^\omega$,
- The subspace $S$ of simple functions is dense in $C(2^\omega)$ (a consequence of Stone-Weierstrass theorem, since $2^\omega$ is totally disconnected)

# The case of $C(2^\omega)$

## Proposition

*The map* $\tilde{\alpha} : C(2^\omega) /\!/ \mathrm{Isom}_{lat}(C(2^\omega)) \to (\mathcal{K}^*(\mathbb{R}), d_H)$, *defined by*

$$\tilde{\alpha}([f]) = \alpha(f) := f(2^\omega),$$

*is a surjective isometry.*

If now we allow the full isometry group, including multiplication by unimodular functions, then we can easily get convinced that the invariant becomes $f \mapsto |f(2^\omega)|$, and so that:

## Proposition

*The map* $\tilde{\beta} : C(2^\omega) /\!/ \mathrm{Isom}(C(2^\omega)) \to (\mathcal{K}^*(\mathbb{R}^+), d_H)$, *defined by*

$$\tilde{\beta}([f]) = \beta(f) := |f(2^\omega)|,$$

*is a surjective isometry.*

# The case of $C(2^\omega)$

## Proposition

*The map* $\tilde\beta : C(2^\omega) /\!/ \mathrm{Isom}(C(2^\omega)) \to (\mathcal{K}^*(\mathbb{R}^+), d_H)$, *defined by*

$$\tilde\beta([f]) = \beta(f) := |f(2^\omega)|,$$

*is a surjective isometry.*

In particular

## Theorem (compactness of the isotropic quotient)

*The isotropic quotient*

$$\mathcal{I}_1(C(2^\omega)) = B(C(2^\omega)) /\!/ \mathrm{Isom}(C(2^\omega))$$

*is isometric to the compact space* $(\mathcal{K}^*([0,1]), d_H)$.

# Proof that $\tilde{\alpha}$ is a surjective isometry

Pf: (1) $\tilde{\alpha}$ defines a bijection from $[S]$ to $\mathcal{F}^*(\mathbb{R})$=$\{$*finite sets* $\}$

*(a) Surjectivity:*
*Proof:* If $F \in \mathcal{F}^*(\mathbb{R})$ has cardinality $n$, then write $2^\omega = \cup_{i=1}^n \mathcal{O}_i$, $\mathcal{O}_i$ clopen, to define a function $f$ with $\alpha(f) = F$.

*(b) Injectivity*:
*Proof:* Assume $f(2^\omega) = g(2^\omega) = \{z_1, \ldots, z_k\}$ and let $F_i := f^{-1}(z_i), G_i := g^{-1}(z_i)$; then all $F_i$ and $G_i$ are clopen and therefore homeomorphic to $2^\omega$; pick any homeomorphism $\phi$ of $2^\omega$ sending $F_i$ onto $G_i$, then $\phi$ will satisfy that $g \circ \phi = f$; therefore $[f] = [g]$.

# Proof that $\tilde{\alpha}$ is a surjective isometry

(1) $\tilde{\alpha}$ induces a bijection between $[S]$ and $\mathcal{F}^*(\mathbb{R})$ OK.

(2) for $f, g \in S$, if $d_H(f(2^\omega), g(2^\omega)) < \varepsilon$, then $d([f],[g]) < \varepsilon$.

Admitting (2) for now, we deduce from (1), (2) and 1-Lipschitzness of $\alpha$, that

- $\tilde{\alpha}$ induces an isometry between $[S]$ and $\mathcal{F}^*(\mathbb{R})$,

and by density of $S$ in $C(2^\omega)$ and of $\mathcal{F}^*(\mathbb{R})$ in $\mathcal{K}^*(\mathbb{R})$, we deduce that

- $\tilde{\alpha}$ is a surjective isometry.

In particular,

$$C(2^\omega) \mathbin{/\!/} \mathrm{Isom}_{\mathrm{lat}}(C(2^\omega)) \equiv \mathcal{K}^*(\mathbb{R}).$$

# (2): $d_H(f(2^\omega), g(2^\omega)) < \varepsilon \Rightarrow d([f],[g]) < \varepsilon, \forall f, g \in S$

Let us prove (2): assume $f, g \in S$ with $d_H(\alpha(f), \alpha(g)) < \varepsilon$. Let $M$ be the finite set

$$M := \{m = (x, y) : x \in \alpha(f), y \in \alpha(g), d(x, y) < \varepsilon\}$$

We note that

- $\forall x \in \alpha(f), \exists y \in \mathbb{R} : (x, y) \in M$
- $\forall y \in \alpha(g), \exists x \in \mathbb{R} : (x, y) \in M$

Now write $2^\omega = \cup_{m \in M} C_m$ a disjoint union of homeomorphic copies of $2^\omega$, and let $f', g'$ be defined by

$$f'_{|C_m} \equiv x, g'_{|C_m} \equiv y, \text{when } m = (x, y).$$

We conclude by noting

- $\|f' - g'\|_\infty < \varepsilon$
- $f'(2^\omega) = \cup_{m=(x,y) \in M}\{x\} = \alpha(f) = f(2^\omega)$
- likewise $g'(2^\omega) = g(2^\omega)$,
- and therefore $[f] = [f']$ and $[g] = [g']$, so $d([f], [g]) < \varepsilon$.

# The case of $C(2^\omega)$: conclusion

Summing up, the map

$$\beta : C(2^\omega) \to \mathcal{K}^*(\mathbb{R}^+)$$

defined by $\beta(f) = |f(2^\omega)|$ induces an isometry between

$$C(2^\omega) \,/\!/\, \mathrm{Isom}(C(2^\omega)) \text{ and } \mathcal{K}^*(\mathbb{R}^+).$$

and between

$$\mathcal{I}_1(C(2^\omega)) = B_{C(2^\omega)} \,/\!/\, \mathrm{Isom}(C(2^\omega)) \text{ and } \mathcal{K}^*([0,1]),$$

and $\mathcal{I}_1(C(2^\omega))$ is compact.

# Spaces with compact isotropic quotients

The isotropic quotient $\mathcal{I}_1(X)$ is compact in the following cases

- $X$ is finite dimensional
- $X$ is almost transitive, so for example $X = L_p(0,1)$ or $X =$ *the Gurarij space* $\mathbb{G}$
- $X = C(2^\omega, \mathbb{R})$.
- other examples?
- also $(X \oplus \ldots \oplus X)_p$, for $X$ as above and $1 \leq p \leq \infty$, but what else?

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- other examples?
- also $(X \oplus \ldots \oplus X)_p$, for $X$ as above and $1 \leq p \leq \infty$, but what else?

# The Banach-Stone isometries on $C(K, Y)$

**Definition**

*Let $Y$ be a Banach space. We denote by $\mathrm{Isom}_{BS}(C(K, Y))$ the group of isometries $T_{\phi,S}$ on $C(K, Y)$ defined by*

$$T_{\phi,S}(f)(k) = S(k).(f \circ \phi)(k),$$

*where $\phi \in \mathrm{Homeo}(K)$, and $k \mapsto S(k)$ is a continuous map from $K$ to $(\mathrm{Isom}(Y), SOT)$*

In certain cases, e.g. when $Y$ is strictly convex (Jerison 50), all isometries on $C(K, Y)$ are Banach-Stone isometries. In any case:

**Observation**

*If $B_{C(K,Y)} /\!/ \mathrm{Isom}_{BS}(C(K, Y))$ is compact, then so is $B_{C(K,Y)} /\!/ \mathrm{Isom}(C(K, Y))$.*

# Isotropic quotients: the case of $C(2^\omega, Y)$

**Proposition**

$B_{C(2^\omega,Y)} /\!/ \mathrm{Isom}_{BS}(C(2^\omega, Y)) \equiv \mathcal{K}^*(B_Y /\!/ \mathrm{Isom}(Y))$.

**Corollary**

*If $Y$ is strictly convex, then $\mathcal{I}_1(C(2^\omega, Y)) \equiv \mathcal{K}^*(\mathcal{I}_1(Y))$.*

**Example**

*If $Y = \mathbb{R}$, since $\mathcal{I}_1(\mathbb{R}) = [0, 1]$, we recover the previous result about $C(2^\omega)$.*

So we can go back to the previous slide and add:

# Spaces with compact isotropic quotients

The isotropic quotient $\mathcal{I}_1(X)$ is compact in the following cases:

- $X$ is finite dimensional
- $X$ is almost transitive, so for example $X = L_p(0,1)$ or $\mathbb{G}$
- $X = C(2^\omega, \mathbb{R})$,

and also when $X = C(2^\omega, Y)$ and $\mathcal{I}_1(Y)$ is compact, so:

- $X = C(2^\omega, F)$, $F$ finite dimensional
- $X = C(2^\omega, L_p)$
- $X = C(2^\omega, \mathbb{G})$

# The $L_p$'s come in the picture

Recall that by Greim - Jamison -Kaminska 1994, if $Y$ is almost transitive and $1 \leq p < \infty$, then $L_p(2^\omega, Y)(\equiv L_p([0,1], Y))$ is almost transitive.

In the same vein we prove:

**Theorem (F. Lopez-Abad 24+)**

*If $\mathcal{I}_1(Y)$ is compact and $1 \leq p < \infty$ then $\mathcal{I}_1(L_p(Y))$ is compact.*

- In the case of $L_p(2^\omega, Y)$, the isotropic quotient may be described as a space of probability Radon measures on $\mathcal{I}_1(Y)$ with the appropriate metric, instead of a space of closed subsets of $\mathcal{I}_1(Y)$ as for $C(2^\omega, Y)$.

# Spaces with compact isotropic quotients

The isotropic quotient $\mathcal{I}_1(X)$ is compact in the following cases:

- $X$ is finite dimensional
- $X$ is almost transitive, so for example $X = L_p(0,1)$ or $\mathbb{G}$
- $X = C(2^\omega, \mathbb{R})$,

and also when $\mathcal{I}_1(Y)$ is compact and $X = C(I, Y)$ or $X = L_p(I, Y)$, where $I$ is finite or $2^\omega$. So

- $X = C(2^\omega, L_1)$ (recall Antônio's talk yesterday)
- $X = L_p([0,1], C(2^\omega, \mathbb{R}))$
- $X = \ell_p^7(L_q([0,1], L_r([0,1], \mathbb{G})))$
- etc...

- V. Ferenczi, J. Lopez Abad, in preparation.
- R. Fleming, R. Jamison, *Isometries on Banach spaces. Vol. 2. Vector-valued functions spaces*, Chapman and Hall Monographs and Surveys in Pure and Applied Mathematics, 138, 2008.
- M. Gromov and V. Milman , *A topological application of the isoperimetric inequality*, Am. J. Math. 105 (4) (1983), 843–854.

Given $n \in \mathbb{N}$, let us look at the diagonal action of $\mathrm{Isom}(X)$ on $X^n$, by $g.(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$, or at its restriction to $B_X^n$.

## Definition

*Let $X$ be a Banach space. The n-th isotropic quotient is $\mathcal{I}_n(X) := B_X^n \,/\!\!/\, \mathrm{Isom}(X)$.*

## Definition

Let $X$ be a Banach space. The following are equivalent for all $n$:

(1) $\mathcal{I}_n(X) = B_X^n /\!/ \mathrm{Isom}(X)$ is compact,

(2) $S_X^n /\!/ \mathrm{Isom}(X)$ is compact,

(3) $\mathrm{Age}_n(X) /\!/ \mathrm{Isom}(X)$ is compact (with $d(F, G) := d_H(B_F, B_G)$ on $\mathrm{Age}(X)$).

When (1)-(3) hold for all $n \in \mathbb{N}$, we say that $X$ is oligomorphic.

## Observation

- *When $X$ is separable, this is equivalent to being $\omega$-categorical, an important notion from model theory.*
- *In the general metric context, the name* "approximately oligomorphic" *is used, rather than our choice of* "oligomorphic"*.*

# Some properties of oligomorphic spaces

Recall that a space $Y$ is finitely representable in a space $X$ if $\mathrm{Age}(Y) \subseteq \overline{\mathrm{Age}}^{BM}(X)$, i.e. for any $F \in \mathrm{Age}(Y)$ and any $\varepsilon > 0$, there is $G \in \mathrm{Age}(X)$ which is $1+\varepsilon$-isomorphic to $F$.

## Proposition

*If $X$ is oligomorphic and infinite dimensional then*

1. *(Khanaki 2020s) any separable space finitely representable in $X$ is isometrically embedded in $X$*
2. *(through Dvoretsky) $X$ contains a copy of the Hilbert space*
3. *if $X = C(K)$ then $K$ is uncountable*

# List of oligomorphic spaces

- finite dimensional spaces
- $L_p(0,1)$, $1 \leq p < \infty$ (Ben Yaacov - Berenstein - Henson - Usvyatsov 08)
- the Gurarij space $\mathbb{G}$ (Ben Yaacov- Henson 17)
- $C(2^\omega)$ (Henson 80-90s)
- $L_p(L_q)$, $1 \leq p, q < \infty$ (Henson-Raynaud 11)

and much more generally

### Theorem (F. Lopez-Abad)

- *$C(K, X)$ whenever $K$ is finite or $2^\omega$ and $X$ is oligomorphic.*
- *$L_p(K, X)$ whenever $K$ is finite or $2^\omega$ and $X$ is oligomorphic.*

### Question

*Find a space $X$ for which $\mathcal{I}_1(X)$ is compact but $\mathcal{I}_n(X)$ is not compact for some $n$.*

- Ben Yaacov - Berenstein - Henson - Usvyatsov, *Model theory for metric structures, in: Model theory with applications to algebra and analysis*. Vol. 2, vol. 350, London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 2008, pp. 315–427.
- T. Tsankov, *Groupes d'automorphismes et leurs actions*, Thèse d'Habilitation à Diriger des Recherches, Paris Diderot 2014.