On Mazur rotations problem and its topological and multidimensional aspects Part 3

Valentin Ferenczi
Universidade de São Paulo

New perspectives in Banach spaces and Banach lattices
CIEM Castro Urdiales, July 8-12, 2024
Joint work with Jordi Lopez Abad, UNED
Fapesp project:
https://geometryofbanachspaces.wordpress.com

Recall from last talk

- Define the isotropic quotient $\mathcal{I}_1(X) := B_X \,/\!/\, \mathrm{Isom}(X)$ (with $d([x],[y]) = \inf \|x - Ty\|$)
- When $X = C(2^\omega)$ then this is compact (with a proof)
- The same holds for $\mathcal{I}_1(C(2^\omega, Y))$ whenever $\mathcal{I}_1(Y)$ is compact (similar methods)
- The same holds for $\mathcal{I}_1(L_p(2^\omega, Y))$ whenever $\mathcal{I}_1(Y)$ is compact (admitted)

Today we multidimensionalize everything....

Adding multidimensionality to the picture....

For any $n \in \mathbb{N}$, let us look at the diagonal action of $\mathrm{Isom}(X)$ on X^n, by $g.(x_1, \ldots, x_n) = (gx_1, \ldots, gx_n)$, or at its restriction to B_X^n, resp. S_X^n.

Definition

Let X be a Banach space. The n-th isotropic quotient is $\mathcal{I}_n(X) := B_X^n \,/\!\!/\, \mathrm{Isom}(X)$.

Big isometry groups: oligomorphic Banach spaces

Definition

Let X be a Banach space. The following are equivalent for all n:

(1) $\mathcal{I}_n(X) = B_X^n \,/\!\!/\, \mathrm{Isom}(X)$ is compact,

(2) $S_X^n \,/\!\!/\, \mathrm{Isom}(X)$ is compact,

(3) $\mathrm{Age}_n(X) \,/\!\!/\, \mathrm{Isom}(X)$ is compact (with $d(E,F) := \inf_{T \in \mathrm{Isom}(X)} d_H(B_E, TB_F)$).

When (1)-(3) hold for all $n \in \mathbb{N}$, we say that X is oligomorphic.

Observation

- *When X is separable, this is equivalent to being ω-categorical, an important notion from model theory.*
- *In the general metric context, the name* "approximately oligomorphic" *is used, rather than our choice of* "oligomorphic"*.*

List of oligomorphic spaces: the usual suspects

- finite dimensional spaces
- $L_p(0,1)$, $1 \leq p < \infty$ (Ben Yaacov - Berenstein - Henson - Usvyatsov 08)
- the Gurarij space $\mathbb{G}$ (Ben Yaacov- Henson 17)
- $C(2^\omega)$ (Henson 80-90s)
- $L_p(L_q)$, $1 \leq p, q < \infty$ (Henson-Raynaud 11)

and much more generally

Theorem (F. - Lopez-Abad 24+)

- *$C(K, X)$ whenever K is finite or 2^ω and X is oligomorphic.*
- *$L_p(K, X)$ whenever K is finite or 2^ω and X is oligomorphic.*

Question

Find a space X for which $\mathcal{I}_1(X)$ is compact but $\mathcal{I}_n(X)$ is not compact for some n.

Why are oligomorphic spaces interesting?

Recall that a space Y is finitely representable in a space X if $\mathrm{Age}(Y) \subseteq \overline{\mathrm{Age}}^{BM}(X)$, i.e. for any $E \in \mathrm{Age}(Y)$ and any $\varepsilon > 0$, there is $F \in \mathrm{Age}(X)$ which is $1 + \varepsilon$-isomorphic to E.

Proposition

If X is oligomorphic and infinite dimensional then

1. *(Khanaki 2020s) any separable space finitely representable in X is isometrically embedded in X*
2. *(through Dvoretsky) X contains a copy of the Hilbert space*
3. *if $X = C(K)$ then K is uncountable*

To say more, let us go back to multidimensional Mazur problem.

(AuH) spaces

Definition (F. LA. M. T. 2020)

A Banach space X is

- AUH *(approximately ultrahomogeneous) if for any* $F \in \mathrm{Age}(X)$, $\varepsilon > 0$, *and isometric embeddings* $i, j : F \to X$, *there exists* $T \in \mathrm{Isom}(X)$ *such that* $\|Ti - j\| \leq \varepsilon$
- *recall that X is* UH *(ultrahomogeneous) if can get rid of* ε

Observation

Let $E, F \in \mathrm{Age}(X)$. *Then:*

- $d(E, F) = 0 \Rightarrow d_{BM}(E, F) = 0$
- *if X is AUH then* $d(E, F) = 0 \Leftrightarrow d_{BM}(E, F) = 0$

This is an instance of position vs shape, i.e. in AUH spaces, fin. dim. subspaces of the same shape are in the same position.

Examples of (AuH) spaces

We already mentioned:

Fact

Any Hilbert space is ultrahomogeneous.

Theorem

Are (AuH)*, but not ultrahomogeneous:*

- *The Gurarij space, defined by Gurarij in 1966 (Kubis-Solecki 2013).*
- $L_p[0,1]$ *for* $p \neq 2, 4, 6, 8, \ldots$ *(Lusky 1978).*

Let us cite Lusky:
"We show that a certain homogeneity property holds for $L_p(0,1)$; $p \neq 4, 6, 8, \ldots$, which is similar to a corresponding property of the Gurarij space..."

Examples of (AuH) spaces

Theorem

Are (AuH)*, but not ultrahomogeneous:*

- *The Gurarij space, defined by Gurarij in 1966 (Kubis-Solecki 2013)*
- $L_p[0,1]$ *for* $p \neq 4, 6, 8, \ldots$ *(Lusky 1978)*

Note that

- the Gurarij is the unique separable, universal, (AuH) space (we may take this as a definition).
- Lusky's result abour L_p's is based on the *equimeasurability theorem* by Plotkin / Rudin, 1976. His proof gives (AuH).
- L_p is not (AuH) for $p = 4, 6, 8, \ldots$:
  B. Randrianantoanina (1999) proved that for those $p's$ there are two isometric subspaces E, F of L_p (due to Rosenthal), well complemented/ badly complemented.

Two problems on separable (AuH) spaces

A possible way towards multidimensional Mazur problem:

Problem

Show that the only separable (AuH) *spaces are the Gurarij and* L_p, $p \neq 4, 6, 8, \ldots$.

Remember that we claimed that the ultrapower $(L_p)_{\mathcal{U}}$ is ultrahomogeneous for $p \neq 4, 6, 8$... Let us explain how this is proved. One would expect this to follow from (AuH), in the same way as the transitivity of $(L_p)_{\mathcal{U}}$ follows from the almost transitivity of L_p. However:

Problem

We do not know whether X $\mathrm{AUH} \Rightarrow X_{\mathcal{U}}$ ultrahomogeneous

We define a formally stronger property of X which does imply that $X_{\mathcal{U}}$ is ultrahomogeneous.

Fraïssé Banach spaces

Given two Banach spaces E and X, and $\delta \geq 0$, let $\mathrm{Emb}_\delta(E, X)$ be the collection of all linear δ-isometric embeddings $T : E \to X$, i.e. such that $\|T\|, \|T^{-1}\| \leq 1 + \delta$ (T^{-1} defined on $T(E)$), equipped with the distance induced by the norm.

We consider the canonical action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$:

Definition (F., Lopez-Abad, Mbombo, Todorcevic 20)

X is Fraïssé if and only if for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there is $\delta = \delta(n, \varepsilon) > 0$ such that for every $E \in \mathrm{Age}_n(X)$, the action $\mathrm{Isom}(X) \curvearrowright \mathrm{Emb}_\delta(E, X)$ has "ε-dense" orbits
(i.e. for every δ-isometric embeddings i, j of E into X, there is $T \in \mathrm{Isom}(X)$ such that $\|j - Ti\| \leq \varepsilon$).

Fraïssé Banach spaces

Definition

X is Fraïssé if and only if for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ there is $\delta = \delta(n, \varepsilon) > 0$ such that for every $E \in \mathrm{Age}_n(X)$, for every δ-isometric embeddings i, j of E into X, there is $T \in \mathrm{Isom}(X)$ such that $\|j - Ti\| \leq \varepsilon$.

Note that:

- Fraïssé $\Rightarrow$ (AuH).
- Hilbert spaces are Fraïssé ($\varepsilon = \delta$, exercise);
- the Gurarij space is Fraïssé (actually $\varepsilon = 2\delta$) ;
- L_p is not Fraïssé for $p = 4, 6, 8, \ldots$ since not (AuH).

Theorem (FLAMT 20)

The spaces $L_p[0,1]$ for $p \neq 4, 6, 8, \ldots$ are Fraïssé.

Theorem (FLAMT 20)

The spaces $L_p[0,1]$ for $p \neq 4,6,8,\ldots$ are Fraïssé.

Proposition

If X is Fraïssé, then its ultrapowers are ultrahomogeneous.

Corollary

The non-separable L_p-space $(L_p(0,1))_{\mathcal{U}}$ is ultrahomogeneous.

We do not know whether $(\mathrm{AuH}) \Rightarrow$ Fraïssé.

Properties of Fraïssé spaces

- We (FLAMT 2020) obtained internal characterizations of classes of finite dimensional spaces which “are” the age of some Fraïssé (”amalgamation properties”). For such a class $\mathcal{F}$ we write X=Fraïssé lim $\mathcal{F}$ to mean ”X separable and $Age(X) \equiv \mathcal{F}$”
- and we also prove a version of the 2005 Kechris-Pestov-Todorcevic correspondance: (X AUH + some approximate Ramsey property) $\Rightarrow \mathrm{Isom}(X)$ is extremely amenable. As a consequence of this + proving a Ramsey property for ℓ_p^n’s, we recover that $\mathrm{Isom}(L_p)$ is extremely amenable (due to Giordano-Pestov 2004)

Properties of Fraïssé spaces

- Cuth, Dolezal, Doucha, Kurka (2022-2023) define a Polish topology on the set of separable spaces and Cuth, de Rancourt, Doucha (2024) show that if X is Fraïssé then X is isometrically generic in the class of Y *finitely representable* in X.
- This implies that separable Fraïssé are isometrically determined, among Fraïssé spaces, by their local structure.
- Cuth, de Rancourt, Doucha (2024+): L_p, for $p = 4, 6, 8 \ldots$, or some $L_p(L_q)$ are examples of the generic property which are not Fraïssé.
- The same authors show that for any separable infinite dimensional oligomorphic space X, there is a generic space $Gen(X)$ with $\overline{\mathrm{Age}}^{BM}(Gen(X)) = \mathrm{Age}(X)$.

Shape versus position

Remember $(\mathrm{Age}_n(X) \mathbin{/\!/} \mathrm{Isom}(X), d)$, where d is the quotient metric $d(E, F) = \inf_{T \in \mathrm{Isom}(X)} d_H(TB_E, B_F)$. We may interpret this space as the metric space of positions of n-dimensional spaces inside X.
On the other hand, $(\mathrm{Age}_n(X) \mathbin{/\!/} \mathrm{Isom}(X), d_{BM})$ is the metric space of shapes of n-dimensional subspaces of X.

Fact

If X is Fraïssé then the map $\tilde{Id} : (\mathrm{Age}_n(X) \mathbin{/\!/} \mathrm{Isom}(X), d) \to (\mathrm{Age}_n(X) \mathbin{/\!/} \mathrm{Isom}(X), d_{BM})$ is a uniform homeomorphism.

So Fraïssé implies that subspaces of “close enough” shapes are in “close enough” positions. While (AuH) implies that subspaces with the same shape are in the same position.

Relation with oligomorphic spaces

Fact

If X is Fraïssé then the map
$\tilde{Id} : (\mathrm{Age}_n(X) \,/\!/\, \mathrm{Isom}(X), d) \to (\mathrm{Age}_n(X) \,/\!/\, \mathrm{Isom}(X), d_{BM})$ *is a uniform homeomorphism.*

Corollary

If X is Fraïssé then X is oligomorphic. In this case the quotient space $\mathrm{Age}_n(X) \,/\!/\, \mathrm{Isom}(X)$ is homeomorphic to the Banach-Mazur compact space $(\mathrm{Age}_n(X), BM)$.

With a bit more work:

Proposition (F. Rincon 2024)

Fraïssé $\Leftrightarrow$ oligomorphic and AUH

(suggested by I. Ben Yaacov + a better proof by F., Lopez-Abad)

Comment on the proof of Fraïsséness for Lps

Theorem (F. LA. M. T. 20)

The Lebesgue spaces $L_p[0,1]$, for $p \notin 2\mathbb{N}+4$, are Fraïssé.

Proof of Theorem

(1) F. LA. M. T.: ”explicit” proof with possible estimate of moduli of uniform continuity.

(2) Abstract proof: L_p-spaces are

- oligomorphic for all $1 \leq p < \infty$ (Ben Yaacov, Berenstein, Henson, Usvyatsov 08)
- and also AUH if $p \neq 4, 6, 8, \ldots$ (Lusky 78),
- Then use F. Rincon 2024, Fraïssé $\Leftrightarrow$ oligomorphic and AUH.

We end with open questions:

Question

Show that every separable ultrahomogeneous space is Hilbertian.

Question

Show that all separable Fraïssé spaces are either isometric to $\mathbb{G}$ or to some L_p.

Proposition (F. Lopez-Abad 24)

Fraïssé spaces with C_∞-bump functions must be isomorphic to a Hilbert space.

Question

Show that (AuH) implies oligomorphic and therefore Fraïssé.

Questions on renormings of classical spaces

Recall that for $p \neq 2$, L_p is not transitive, and ℓ_p not almost transitive. Furthermore

Theorem (Dilworth - Randrianantoanina, 2014)

Let $1 < p < +\infty, p \neq 2$. Then
ℓ_p does not admit an equivalent almost transitive norm.

(a first very exotic superreflexive example with no almost transitive renorming had been obtained by F. - Rosendal, 2013)

Question

Let $1 \leq p < +\infty, p \neq 2$. Show that the space $L_p(0,1)$ does not admit an equivalent transitive norm. Same question for ultrahomogeneous renorming.

Proposition (F. Lopez-Abad 24)

- *L_4, etc... does not admit a Fraïssé renorming.*
- *L_1 does not admit a transitive Fraïssé renorming.*

Question

Show that any Fraïssé renorming of L_p is isometric to L_p (open even for $p = 2$)

Question

Are the Hilbert and the Gurarij the only stable separable Fraïssé spaces (Fraïssé property independent of the dimension)? Are the $L_p(0,1)$ spaces stable Fraïssé for p non even?

- M. Cúth, N. de Rancourt, M. Doucha, *Guarded Fraïssé Banach spaces*, in preparation 2024.
- V. F., J. Lopez Abad, B. Mbombo, S. Todorcevic, *Amalgamation and Ramsey properties of L_p-spaces*, Advances in Math. 369 (2020), 107190.
- V. F., J. Lopez-Abad, *Envelopes in Banach spaces*. 2024.
- A. S. Kechris, V. G. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups.* Geo. Func. An. **15** (2005) 106–189.
- V. Pestov, *Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon.* Revised edition of [IMPA, Rio de Janeiro, 2005]. University Lecture Series, 40. AMS, Providence, RI, 2006.
- T. Tsankov, *Groupes d'automorphismes et leurs actions*, Thèse d'Habilitation à Diriger des Recherches, Paris 7 (2014)

THANK YOU - GRACIAS!