Asymptotic smoothness and concentration properties

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Hamming graphs

Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$.

We denote

$$[\mathbb{M}]^\omega = \{S \subset \mathbb{M}; S \text{ is infinite}\};$$

$$[\mathbb{M}]^k = \{\overline{n} = (n_1, \cdots, n_k) \in \mathbb{M}^k; n_1 < \cdots < n_k\};$$

$$[\mathbb{M}]^{\leq k} = \bigcup_{i=1}^{k} [\mathbb{M}]^i.$$

Hamming graphs

We equip $[\mathbb{N}]^k$ with the Hamming distance:

$$d_{\mathbb{H}}(\overline{n},\overline{m}) = |\{j;\, n_j \neq m_j\}|$$

for all $\overline{n} = (n_1, \cdots, n_k)$, $\overline{m} = (m_1, \cdots, m_k) \in [\mathbb{N}]^k$.

Note. It is a graph distance.

Consequence

Let $(X, \|.\|)$ be a Banach space and $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ a Lipschitz map.

$$\mathrm{Lip}(f) = \sup_{d_{\mathbb{H}}(\overline{n},\overline{m})=1} \|f(\overline{n}) - f(\overline{m})\| = \max_{1\leq j\leq k} \underbrace{\sup_{\substack{d_{\mathbb{H}}(\overline{n},\overline{m})=1\\ n_j\neq m_j}} \|f(\overline{n}) - f(\overline{m})\|}_{\mathrm{Lip}_j(f)}$$

Asymptotic uniform smoothness

Definitions

• A tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset X$ is said to be weakly null if the sequence $(x_{\overline{n},n})_{n>\max(\overline{n})}$ is weakly null for every $\overline{n} \in [\mathbb{N}]^{\leq k-1} \cup \{\varnothing\}$ (with $\max(\varnothing) = 0$).

• X is said to have A_p, $1 < p \leq \infty$ if there exists $\lambda > 0$ st for any $k \in \mathbb{N}$ and any weakly null tree $(x_{\overline{n}})_{\overline{n}\in[\mathbb{N}]^{\leq k}} \subset B_X$, we can find $\mathbb{M} \in [\mathbb{N}]^{\omega}$ st

$$\forall \overline{n} \in [\mathbb{M}]^k,\ \forall a = (a_i)_{i=1}^k \in \mathbb{R}^k,\ \Big\| \sum_{j=1}^{k} a_j x_{n_1,\cdots,n_j} \Big\| \leq \lambda \|a\|_{\ell_p^k}.$$

• X is asymptotically uniformly smooth up to renorming (AUSable) if X has A_p for some $1 < p \leq \infty$.

Example

• If (F_n) is a sequence of finite dimensional spaces, $\left(\sum_{n\in\mathbb{N}} F_n\right)_{\ell_p}$ has A_p;

• c_0 has A_∞.

Previous results

Theorem (Kalton-Randrianarivony '08)

Let $1 < p < \infty$ and X be a reflexive Banach space with A_p.
Then there exists $\lambda > 0$ such that for every $k \in \mathbb{N}$, for every Lipschitz map $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that

$$\forall \overline{n}, \overline{m} \in [\mathbb{M}]^k,\ \|f(\overline{n}) - f(\overline{m})\| \leq \lambda \Big(\sum_{j=1}^{k} \mathrm{Lip}_j(f)^p \Big)^{1/p}.$$

$\rightsquigarrow$ X has property λ-$\mathsf{HFC}_{p,d}$.

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$\rightsquigarrow$ X has property λ-$\mathrm{HFC}_{p,d}$.

Theorem (Baudier, Lancien, Motakis, Schlumprecht '21)

Let X be a reflexive Banach space with A_∞.
Then there exists $\lambda > 0$ such that for every $k \in \mathbb{N}$, for every Lipschitz map $f : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ so that

$$\forall \overline{n}, \overline{m} \in [\mathbb{M}]^k,\ \|f(\overline{n}) - f(\overline{m})\| \leq \lambda \,\mathrm{Lip}(f).$$

$\rightsquigarrow$ X has property λ-HFC_∞.

Some remarks

Remarks

1) X HFC$_{p,d}$ $\implies$ $([\mathbb{N}]^k, d_{\mathbb{H}}) \underset{eL}{\not\hookrightarrow} X$

where $([\mathbb{N}]^k, d_{\mathbb{H}}) \underset{eL}{\hookrightarrow} X$: there exist $A, B > 0$ and $(f_k : [\mathbb{N}]^k \to X)_k$ such that, for every $k \in \mathbb{N}$

$$\forall \overline{n}, \overline{m} \in [\mathbb{N}]^k,\ A d_{\mathbb{H}}(\overline{n}, \overline{m}) \leq \|f(\overline{n}) - f(\overline{m})\| \leq B d_{\mathbb{H}}(\overline{n}, \overline{m}).$$

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1) X HFC$_{p,d}$ $\implies$ $([\mathbb{N}]^k, d_{\mathbb{H}}) \underset{eL}{\not\hookrightarrow} X$

2) $X \underset{cL}{\hookrightarrow} Y$ HFC$_{p,d}$ $\implies$ X HFC$_{p,d}$

where $X \underset{cL}{\hookrightarrow} Y$: there exist $f : X \to Y$ and $A, B, \theta > 0$ such that

$$\forall x, x' \in X,\ \|x - x'\| \geq \theta \implies A\|x - x'\| \leq \|f(x) - f(x')\| \leq B\|x - x'\|.$$

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2) $X \underset{cL}{\hookrightarrow} Y$ $\mathrm{HFC}_{p,d} \implies X$ $\mathrm{HFC}_{p,d}$

Note. 2) provides an obstruction to the coarse-Lipschitz embedding of certain spaces into Y $\mathrm{HFC}_{p,d}$.

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Example. For every $1 \leq q < p$, $\ell_q \underset{cL}{\not\hookrightarrow} \ell_p$

$$\left(f : \begin{cases} [\mathbb{N}]^k & \to \quad \ell_q \\ \overline{n} & \mapsto \quad \sum_{j=1}^k e_{n_j} \end{cases} \right)$$

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3) HFC$_{p,d}$ $\implies$ reflexivity

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3) HFC$_{p,d}$ $\implies$ reflexivity

Note. $X \underset{cL}{\hookrightarrow} Y$ reflexive AUSable $\implies$ X reflexive

Main question

Open question (Godefroy)

$X \underset{cL}{\hookrightarrow} Y$ reflexive AUSable $\implies$ X AUSable?

Question

$\mathrm{HFC}_{p,d} \implies$ AUSable ($i.e$ A_p for some $p > 1$)?

Theorem (Baudier, Lancien, Motakis, Schlumprecht '21)

$HFC_\infty \implies \mathrm{A}_\infty$.

$\mathrm{HFC}_{p,d}$ does not imply AUSable

Proposition

The space X_ω is not AUSable.

If we prove

Lemma

Let $\lambda > 0$, $p \in (1, \infty)$ and X be a Banach space with a finite codimensional subspace Y that has property λ-$\mathrm{HFC}_{p,d}$. Then X has $(\lambda + \varepsilon)$-$\mathrm{HFC}_{p,d}$ for every $\varepsilon > 0$.

Theorem

Let $p \in (1, \infty)$, $\lambda > 0$, $(X_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces with property λ-$HFC_{p,d}$.
Then $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_{\ell_p}$ has property $(\lambda + \varepsilon)$-$HFC_{p,d}$ for every $\varepsilon > 0$.

$$X_0 = \mathbb{R} \oplus_1 \ell_2,$$

$$\forall n \in \mathbb{N},\ X_{n+1} = \mathbb{R} \oplus_1 \ell_2(X_n),$$

$$X_\omega = \Big(\sum_{n\in\mathbb{N}} X_n \Big)_{\ell_2}.$$

Then

Corollary

The space X_ω has property $\mathrm{HFC}_{2,d}$ without being AUSable.

$\mathrm{HFC}_{p,d}$ does not imply AUSable

Theorem

Let $p \in (1, \infty)$, $\lambda > 0$, $(X_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces with property λ-$\mathrm{HFC}_{p,d}$.
Then $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_{\ell_p}$ has property $(\lambda+\varepsilon)$-$\mathrm{HFC}_{p,d}$ for every $\varepsilon > 0$.

Note.

$\left(\sum_{n\in\mathbb{N}} X_n\right)_{\ell_p} = \left\{x = (x_n); \forall n, x_n \in X_n, \|x\| = \left(\sum_{n\in\mathbb{N}} \|x_n\|_{X_n}^p\right)^{1/p} < \infty\right\}$.

$X_n = X \rightsquigarrow \ell_p(X)$.

A few words about the proof

Proposition

Let X_1 and X_2 be two Banach spaces with λ-HFC$_{p,d}$.
Then $X = X_1 \oplus_p X_2$ has $(\lambda + \varepsilon)$-HFC$_{p,d}$ for every $\varepsilon > 0$.

It follows directly from the following lemma:

Lemma

Let $p \in (1, +\infty)$, X_1 and X_2 two Banach spaces, $X = X_1 \bigoplus_p X_2$, $k \in \mathbb{N}$.

For every $\varepsilon > 0$ and every Lipschitz map $h = (f, g) : [\mathbb{N}]^k \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

$$\mathrm{Lip}_j(f_{|[\mathbb{M}]^k})^p + \mathrm{Lip}_j(g_{|[\mathbb{M}]^k})^p \leq \mathrm{Lip}_j(h)^p + \varepsilon$$

for every $1 \leq j \leq k$.

Lemma

Let $p \in (1, +\infty)$, X_1 and X_2 two Banach spaces, $X = X_1 \bigoplus_p X_2$, $k \in \mathbb{N}$.

For every $\varepsilon > 0$ and every Lipschitz map $h = (f, g) : [\mathbb{N}]^k \to X$, there exists $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

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for every $1 \leq j \leq k$.

Ramsey's Theorem

Let F be a finite set and $\phi : [\mathbb{N}]^k \to F$.
Then there exist $f \in F$ and $\mathbb{M} \in [\mathbb{N}]^\omega$ such that $\phi(\overline{n}) = f$ for all $\overline{n} \in [\mathbb{M}]^k$.

$$\mathrm{Lip}_j(f) = \sup_{(\overline{n},\overline{m}) \in H_j(\mathbb{N})} \|f(\overline{n}) - f(\overline{m})\|$$

$$H_j(\mathbb{M}) = \{(\overline{n}, \overline{m}) \subset [\mathbb{M}]^k ; d_{\mathbb{H}}(\overline{n}, \overline{m}) = 1, n_j < m_j\} \longleftrightarrow [\mathbb{M}]^{k+1}$$

Theorem

Let $p \in (1,\infty)$, $\lambda \geq 2$, $(X_n)_{n\in\mathbb{N}}$ a sequence of Banach spaces with property λ-$HFC_{p,d}$.
Then $X = \left(\sum_{n\in\mathbb{N}} X_n\right)_{\ell_p}$ has property $(\lambda+\varepsilon)$-$HFC_{p,d}$ for every $\varepsilon > 0$.

Sketch of proof

Let $f = (f_n)_n : ([\mathbb{N}]^k, d_{\mathbb{H}}) \to X$ Lipschitz. The map

$$\phi : \begin{cases} X & \to & \ell_p \\ (x_n) & \mapsto & (\|x_n\|) \end{cases}$$

is 1-Lipschitz, $\mathrm{Lip}_j(\phi \circ f) \leq \mathrm{Lip}_j(f)$.
$\phi \circ f : [\mathbb{N}]^k \to \ell_p$. Proof of KR $\rightsquigarrow$ there exist $u \in \ell_p$, $\mathbb{M} \in [\mathbb{N}]^\omega$ such that

$$\forall \overline{n} \in [\mathbb{M}]^k,\ \|\phi \circ f(\overline{n}) - u\| \leq \Big(\sum_{j=1}^{k} \mathrm{Lip}_j(f)^p\Big)^{\frac{1}{p}} + \varepsilon.$$

Take N such that $\left(\sum_{j=N+1}^{\infty} |u_j|^p\right)^{1/p} \leq \varepsilon$. Note $\Pi_N : X \to \left(\sum_{n=1}^{N} X_n\right)_{\ell_p}$.

Proof of the theorem

Proposition $\rightsquigarrow \mathbb{M}' \in [\mathbb{M}]^\omega$ such that:

$$\|\Pi_N \circ f(\overline{n}) - \Pi_N \circ f(\overline{m})\| \leq (\lambda + \varepsilon)\Big(\sum_{j=1}^{k} \mathrm{Lip}_j(\Pi_N \circ f)^p\Big)^{\frac{1}{p}}$$

for all $\overline{n}, \overline{m} \in [\mathbb{M}']^k$.
Proof of KR $\rightsquigarrow$ there exists $\mathbb{M}'' \in [\mathbb{M}']^\omega$ such that

$$\forall \overline{n} \in [\mathbb{M}'']^k,\ \|\phi \circ (I - \Pi_N) \circ f(\overline{n}) - v\| \leq \Big(\sum_{j=1}^{k} \mathrm{Lip}_j((I - \Pi_N) \circ f)^p\Big)^{\frac{1}{p}} + \varepsilon$$

where $v = \sum_{j=N+1}^{\infty} u_j e_j \in \ell_p$.

Conclude by applying the lemma one more time, using that $\|v\| \leq \varepsilon$.

A remark

Let X be a separable Banach space. $Sz(X)$: ordinal quantification of its Asplundness (a separable Banach space is Asplund iff it has separable dual), or of how close to be non separable X^* is.

Theorem

(i) $Sz(X) < \omega_1 \iff X^*$ is separable;
(ii) $Sz(X) \leq \omega \iff X$ is AUSable.

Theorem

For every ordinal $\alpha < \omega_1$ and every $1 < p < \infty$, there exists a Banach space X with property $\mathrm{HFC}_{p,d}$ and $Sz(X) > \alpha$.

<u>Note:</u> The set of spaces with separable dual satisfying $\mathrm{HFC}_{p,d}$, $1 < p < \infty$, is not Borel.

Open questions

1) $X \underset{cL}{\hookrightarrow} Y$ reflexive AUS $\overset{?}{\Longrightarrow}$ X AUSable

2) Characterization of the spaces that equi-Lipschitz contain the Hamming graphs?

Thanks for listening!