On the complete separation of unique ℓ_1 spreading models and the Lebesgue property

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The Lebesgue property

A Banach space X is said to have the **Lebesgue property** (LP) if every Riemann-integrable (RI) function $f : [0, 1] \to X$ is Lebesgue almost-everywhere continuous.

(i) All classical Banach spaces except for ℓ_1 do not have the LP.

(ii) Tsirelson's space has the LP and, more generally, every asymptotic-ℓ_1 Banach space has the LP.

In [GS], the LP is characterized in terms of a new sequential asymptotic structure.

The Lebesgue property as an asymptotic structure

A collection $\{A_i^n\}_{i=1,n\in\mathbb{N}}^{2^n}$ of infinite subsets of $\mathbb{N}$ is said to be a **Haar system** if

- $\mathbb{N} = \cup_{i=1}^{2^n} A_i^n$ and $A_i^n \cap A_{i'}^n = \emptyset$ if $i \neq i'$.
- $A_i^n = A_{2i-1}^{n+1} \cup A_{2i}^{n+1}$ for each $n \in \mathbb{N}$ and for each $1 \leq i \leq 2^n$.

Then, a normalized basic sequence (b_j) in X is said to be **Haar-ℓ_1^+** if, for every Haar system $\{A_i^n\}_{i=1,n\in\mathbb{N}}^{2^n}$,

$$0 < \lim_{n\to\infty} \sup \left\{ \frac{1}{2^m} \left\| \sum_{k=1}^{2^m} b_{j_k} \right\| \;\middle|\; m \geq n \text{ and } 2^m \leq j_k \in A_k^m \right\}.$$

It is proved in [GS] that a Banach space X has the LP if and only if every normalized basic sequence in X is Haar-ℓ_1^+.

The Banach space X_{iw} - preliminaries

Let (m_j) and (n_j) be two strictly increasing sequences of real numbers (that satisfy certain technical requirements) and recall that the Schreier sets are an increasing sequence of collections of finite subsets of $\mathbb{N}$ that are defined inductively by

$$S_0 = \{\{i\} \mid i \in \mathbb{N}\} \text{ and } S_1 = \{F \subset \mathbb{N} \mid |F| \leq \min(F)\}$$

where $|F|$ denotes the cardinality of F and, for every $n \in \mathbb{N}$, by

$$S_{n+1} = \left\{ F \subset \mathbb{N} \;\middle|\; \substack{F=\bigcup_{i=1}^{d} F_i, \text{ where } F_1<\ldots<F_d, \\ F_i \in S_n \text{ for each } i, \text{ and } d \leq \min(F_1)} \right\}$$

once S_n has been defined.

The Banach space X_{iw} - definition

Let $W_0 = \{\pm e_i^*\} \subset c_{00}$ and define $w(\pm e_i^*) = \infty$. Then, we may define inductively the set

$$W_{n+1} = W_n \cup \left\{ \frac{1}{m_{j_1} \cdots m_{j_l}} \sum_{q=1}^{d} f_q \;\middle|\; \begin{array}{c} f_1 < \ldots < f_d \text{ are members of } W_n \\ \max\operatorname{supp}(f_{q-1}) < w(f_q) \text{ for each } 2 \leq q \leq d \\ \{\min\operatorname{supp}(f_q)\}_{q=1}^{d} \in S_{n_{j_1}+\cdots+n_{j_l}} \end{array} \right\}$$

once W_n has been defined. We now put $W = \bigcup_{n=0}^{\infty} W_n$ and we define X_{iw} to be the completion of c_{00} with respect to the norm that is given by $\|x\|_W = \sup\{f(x) \mid f \in W\}$, $x \in c_{00}$.

The Banach space X_{iw} - basic properties

The Banach space X_{iw} is reminiscient other Tsirelson-type spaces (e.g. Schlumprecht's space) that are built using norming sets of carefully-chosen weighted functionals, so it is not a surprise that the unit vector basis (e_i) for X_{iw} is 1-unconditional. Moreover,

(i) Every spreading model of X_{iw} that is generated by a normalized block basis of (e_i) is 4-equivalent to the unit vector basis of ℓ_1.
- Enough functionals in W to norm "n after n" vectors

(ii) Every block subspace of X_{iw} contains an infinite array of block sequences of (e_i) that generates a c_0-asymptotic model.
- Array is constructed so that diagonals are "exact" vectors

Lastly, the above two facts and the unconditionality of (e_i) imply that X_{iw} is reflexive.

The Banach space X_{iw} has the LP - 1

It is actually enough to show that every normalized block basis (b_j) of (e_i) is Haar-ℓ_1^+. To that end, let (b_j) be such a block basis and choose $f_j \in W$ so that $f_j(b_j) = 1$ and so that $\mathrm{supp}(f_j) \subset \mathrm{supp}(b_j)$.

Next, let D be the set of dyadic rationals in $[0,1)$, let $\{A_i^n\}_{i=1,n\in\mathbb{N}}^{2^n}$ be a Haar system, and write $D \cap \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) = \{d_j\}_{j\in A_i^n}$.

Case 1: $\sup_{j\in\mathbb{N}} w(f_j) = L < \infty$

Let $s \in \Lambda = [0,1] \setminus D$ and choose $d_j^s \to s$. It follows by passing to a subsequence if needed that the corresponding sequence $w(f_j^s)$ of weights is constant. Thus, put $\Lambda_h = \{s \in \Lambda \mid w(f_j^s) = h\}$ for each $h \in \mathbb{N} \cap [1, L]$ and observe that $\mu^*(\Lambda_{h_0}) > 0$ for at least one h_0.

The Banach space X_{iw} has the LP - 2

Next, let $n \in \mathbb{N}$, define the set

$$I_n = \left\{ i \;\middle|\; \mu^* \left(\left(\frac{i-1}{2^n}, \frac{i}{2^n} \right) \cap \Lambda_{h_0} \right) > 0 \right\} = \{i_1, \ldots, i_r\},$$

and choose for each $1 \leq k \leq r$ a point $s_k \in \left(\frac{i_k - 1}{2^n}, \frac{i_k}{2^n} \right) \cap \Lambda_{h_0}$ and a dyadic rational $2^n \leq d_{j_k}^{s_k} \in A_{i_k}^n$ so that

(i) $b_{j_1} < \ldots < b_{j_r}$ are corresponding block vectors that satisfy $\{\min \operatorname{supp}(b_{j_k})\}_{k=1}^r \in S_{h_0}$.

(ii) The corresponding functional weights, $w(f_{j_k}^{s_k})$, are equal to h_0.

The Banach space X_{iw} has the LP - 3

It is not difficult to verify that $F = \frac{1}{h_0 m_1} \sum_{k=1}^{r} \sum_{q=2}^{d_k} f_q^k \in W$ where $f_{j_k}^{s_k} = \frac{1}{h_0} \sum_{q=1}^{d_k} f_q^k$, and that

$$\frac{1}{2^n} \left\| \sum_{k=1}^{r} b_{j_k} \right\| \geq \frac{1}{2^n} F\left(\sum_{k=1}^{r} b_{j_k} \right) \geq \frac{r}{2m_1 \cdot 2^n} \geq \frac{\mu^*(\Lambda_{h_0})}{2m_1}$$

from which it follows that (b_j) is Haar-ℓ_1^+.

Case 2: $\sup_{j\in\mathbb{N}} w(f_j) = \infty$

This case is a similar idea but actually easier.

Motivation for the definition of $X_{\mathcal{D}}$

A similar argument shows that every spreading model of X_{iw} that is generated by a normalized block basis of (e_i) is equivalent to the unit vector basis of ℓ_1. However, this argument requires only the stabilization of norming functional weights - not the stabilization of norming functional weights *with respect to a subset of* $[0, 1]$ *that has positive Lebesgue outer measure*.

There are enough functionals in W that this extra stabilization is okay for X_{iw}. Thus, we want to define a new norming set, say $\tilde{W}$, which does not allow for this extra stabilization.

The solution: an additional norming set constraint

The dyadic tree $\mathcal{D} = \cup_{n=0}^{\infty}\{0,1\}^n$ is a totally bounded metric space with respect to $d(\lambda,\mu) = 2^{-n}$ where n is the initial height at which there exists $\nu \in \{0,1\}^n$ such that $\nu \leq \lambda$ and $\nu \leq \mu$.

We associate functional weights to nodes of $\mathcal{D}$ and we impose the additional metric contstraint that for functionals of the form

$$f = \frac{1}{w(f)} \sum_{q=1}^{r} f_q \in \tilde{W}$$

the nodes of $\mathcal{D}$ that correspond to $f_1 < \ldots < f_r$ must approximate some $\lambda \in \overline{\mathcal{D}}$. The quantification of this approximation depends on both the admissibility and the supports of $f_1 < \ldots < f_r$.

The Banach space $X_{\mathcal{D}}$

The $\mathcal{D}$-proximity constraint disallows the LP because, in this case, we cannot stabilize with respect to a subset of $[0, 1]$ that has positive outer measure both the weights of norming functionals **and** the correct $\mathcal{D}$-proximity in order to combine them as before.

We therefore define $X_{\mathcal{D}}$ to be the completion of c_{00} with respect to the norming set $\tilde{W}$ that consists of $\pm e_i^*$ and functionals of the form

$$f = \frac{1}{w(f)} \sum_{q=1}^{d} f_q$$

where $f_1 < \ldots < f_d \in \tilde{W}$ satisfy certain increasing weight/support and Schreier admissibility conditions as before, and where also the additional $\mathcal{D}$-proximity condition is satisfied.

$X_{\mathcal{D}}$ fails the LP in every infinite-dimensional subspace

Intuitively, $X_{\mathcal{D}}$ fails the LP in every infinite-dimensional subspace because the $\mathcal{D}$-proximity condition disallows the required proof in any such subspace.

More formally, we use standard RIS techniques to build in every block subspace a normalized block basis (x_{m_j}) of exact vectors (i.e. x_{m_j} can be normed by a functional of weight m_j only).

We define a Haar system so that the nodes of $\mathcal{D}$ that correspond to the weights of the norming functionals for $x_{m_1} < \ldots < x_{m_{2^n}}$ chosen as in the Haar-ℓ_1^+ condition form a 2^{-n}-separated set. It follows that a functional $f = \frac{1}{w(f)} \sum_{q=1}^{r} f_q \in \tilde{W}$ norms at most one of these vectors and, thus, (x_{m_j}) cannot be Haar-ℓ_1^+.

Additional properties of $X_{\mathcal{D}}$

The proofs for $X_{\mathcal{D}}$ required substantial modification from earlier similar arguments for X_{iw} because of the complexity of our norming set. However, in addition to the failure of the LP in every subspace, we have that

(i) $X_{\mathcal{D}}$ has a uniformly unique ℓ_1 spreading model.

(ii) $X_{\mathcal{D}}$ has a 1-unconditional normalized basis (and thus, $X_{\mathcal{D}}$ is reflexive because it cannot contain c_0 and, since it fails the LP in every subspace, it cannot contain ℓ_1 either).

We therefore arrive at the following theorem.

Theorem (G., Motakis, Sari)

$X_{\mathcal{D}}$ completely separates the LP from a (uniformly) unique ℓ_1 spreading model.