Approximate Morse-Sard result in nonseparable Banach spaces

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In this talk I will present some results contained in the recent paper:

• D. Azagra, M. García-Bravo, M. Jiménez-Sevilla, *Approximate Morse-Sard type results for non-separable Banach spaces*, J. Funct. Anal. 287, no. 4 (2024).

The structure of the talk is as follows:

1. Introduction to the Morse-Sard theorem in infinite dimensions.
2. Approximate Morse-Sard results in the nonseparable case.
3. Main ideas and tools behind the proofs.

We begin with

1. Introduction to the Morse-Sard theorem in infinite dimensions.

Critical point

Definition (Critical point)

If we have a function $f : E \longrightarrow F$ between Banach spaces which is (Fréchet) differentiable at some point x we will say that x is a critical point if $Df(x) \in \mathcal{L}(E, F)$ is not a surjective operator.

- C_f = set of critical points.
- $f(C_f)$ = set of critical values.

Recall that the (Fréchet) derivative $Df(x)$ of f at x is defined as the unique linear continuous operator such that

$$\lim_{h \to 0} \frac{||f(x+h) - f(x) - Df(x)(h)||}{||h||} = 0.$$

Question: Which regularity conditions do we have to impose to f so that $f(C_f)$ is *small* in some sense?

Theorem (Morse 1939, Sard 1942)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^k function with $k \geq \max\{n - m + 1, 1\}$. Then the set of critical values, $f(C_f)$, is of Lebesgue measure zero in $\mathbb{R}^m$ ($\mathcal{L}^m(f(C_f)) = 0$).

Note that here

$$C_f = \{x \in \mathbb{R}^n : \, rank\, Df(x) < \min\{n, m\}\}.$$

• This result has been shown to be sharp in the class of functions C^j thanks to the famous counterexample of Whitney in 1935. He built a function $f : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that $\mathcal{L}^1(f(C_f)) > 0$.

• This theorem has been generalized to other function spaces such as Hölder spaces $C^{k-1,1}$, Sobolev spaces $W^{k,p}$ with $p > n$ or to the space of functions of bounded variation BV_n.

Infinite dimension: $\dim(E) = \infty$, $\dim(F) = 1$

Kupka's counterexample (1965): There exist C^∞ functions $f : \ell_2 \to \mathbb{R}$ so that their sets of critical values $f(C_f)$ contain intervals.

Example (Bates and Moreira, 2001)

Let $f : \ell_2 \to \mathbb{R}$ be defined as

$$f\left(\sum_{n\geq 1} x_n e_n\right) = \sum_{n\geq 1}(3 \cdot 2^{-\frac{n}{3}} x_n^2 - 2x_n^3)$$

- f is C^∞ (a polynomial of degree three).
- $C_f = \left\{\sum_{n\geq 1} x_n e_n \,:\, x_n \in \{0, 2^{-\frac{n}{3}}\}\right\}$.
- $f(C_f) = [0, 1]$.

Conclusion: We cannot expect to have a good version of the Morse-Sard theorem for infinite dimensions !!

However around 20 years ago the following two results appeared, which can be considered as *approximate Morse-Sard results*.

Theorem

1. **(Azagra, Cepedello 2004)** *For every continuous functions $f : \ell_2 \to \mathbb{R}$ and $\varepsilon : \ell_2 \to (0, \infty)$, there exists a C^∞ function $g : \ell_2 \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in E$ and $C_g = \emptyset$.*
2. **(Azagra, Jiménez-Sevilla 2007)** *Let E be a separable infinite dimensional Banach space with separable dual. Then for every continuous functions $f : E \to \mathbb{R}$ and $\varepsilon : E \to (0, \infty)$, there exists a C^1 function $g : E \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in E$ and $C_g = \emptyset$.*

Approximate Morse-Sard for $\dim(E) = \dim(F) = \infty$

Theorem (Azagra, Dobrowolski, G-B (2019))

Let E be one of the classical Banach spaces c_0, ℓ_p or L^p, $1 < p < \infty$ and let F a (non zero) quotient of E (so there exists a bounded linear surjective operator from E onto F).
Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to F$ *is a surjective linear operator for every* $x \in E$.

Here k denotes the order of smoothness of the norm of the space E.

Warning: The previous results do not hold in finite dimensions.

Related topic: Failure of Rolle's theorem

The previous results are related with the failure of Rolle's theorem in infinite dimensions. This was first noticed by Shkarin in 1992 for superreflexive spaces and non-reflexive spaces with smooth norms.

Theorem (Azagra, Jiménez-Sevilla (2001))

For every infinite-dimensional Banach space E with a C^1 smooth bump there exists another C^1 smooth bump $b : E \to \mathbb{R}$ so that $b'(x) \neq 0$ for every $x \in int(supp(b))$.

We continue now with

2. Approximate Morse-Sard results in the nonseparable case.

Question: Can the previous results be extended to the case of nonseparable Banach spaces?

Theorem (Azagra, G-B, Jiménez-Sevilla (2024))

Let Γ be an arbitrary infinite set, let $E = c_0(\Gamma), \ell_p(\Gamma)$, $1 < p < \infty$ and let F any (non zero) quotient of E. Then for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to F$ *is a surjective linear operator for every* $x \in E$.

Here k denotes the order of smoothness of the norm of the space E.

Definitions:

$$c_0(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \forall\, \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma : |x_\gamma| \geq \varepsilon\} \text{ is finite}\}$$

$$\ell_p(\Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \subset \mathbb{R}^\Gamma : \sum_{\gamma\in\Gamma} |x_\gamma| p < \infty\}$$

Other different versions:

Theorem (Azagra, G-B, Jiménez-Sevilla (2024))

Let E be a Banach space with C^1 partitions of unity. Assume moreover that E contains an infinite-dimensional separable complemented subspace Y. Then for every continuous mapping $f : E \to \mathbb{R}^m$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to \mathbb{R}^m$ such that

1. $\|f(x) - g(x)\| \leq \varepsilon(x)$ *for every* $x \in E$, *and*
2. $Dg(x) : E \to \mathbb{R}^m$ *is a surjective linear operator for every* $x \in E$.

Remark: For this last theorem, in the case that Y cannot be taken to be reflexive, we need to perform an adequate renorming of the space Y.

Main tools of the proofs

We finally move to

3. Main ideas and tools behind the proofs.

We will distinguish between different cases:

1. Case of $c_0(\Gamma)$.
2. Case of $\ell_p(\Gamma)$, $1 < p < \infty$.

Case $c_0(\Gamma)$

The approximating function g has the form

$$g(x) = \sum_{\alpha \in A} \psi_\alpha(x)(f(x_\alpha) + T(x - x_\alpha))$$

1. $\{\psi_\alpha\}_{\alpha \in A}$ is a C^∞ smooth partition of unity subordinate to some open covering $\{U_\alpha\}_{\alpha \in A}$ with $x_\alpha \in U_\alpha$ that has locally the form

$$x \to \psi_\alpha(x) = \varphi_\alpha(e^*_{\gamma_1}(x), \dots, e^*_{\gamma_n}(x)).$$

2. $T : c_0(\Gamma) \to F$ is a continuous surjective operator such that $T|_{c_0(\Gamma_n)}$ is surjective and

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n, \quad \#(\Gamma_n) = \#(\Gamma).$$

For all $x \in c_0(\Gamma)$ we have $\|f(x) - g(x)\| \leq \varepsilon(x)$ and

$$Dg(x) = T + L, \quad L \in \text{span}\{e^*_\gamma : \gamma \in \Gamma\} \quad \Rightarrow \quad Dg(x) \text{ is surjective.}$$

Case $\ell_p(\Gamma)$

We first define

$$p(x) = \sum_{\alpha \in A} \psi_\alpha(x)(f(x_\alpha) + T(x - x_\alpha))$$

in a similar way as before but now we have that $\{\psi_\alpha\}$ is locally of the form

$$x \to \psi_\alpha(x) = \varphi_\alpha(\|x\|^p, e^*_{\gamma_1}(x), \ldots, e^*_{\gamma_n}(x)).$$

The previous comes from the existence of certain homeomorphic embeddings from $\ell_p(\Gamma)$ into $c_0(\Gamma')$ (for some infinite set Γ') whose coordinate functions are C^k smooth, given by Toruńczyk in 1973.

In this case $C_p \neq \emptyset$, but we solve this situation by building a diffeomorphism $h : E \to E \setminus C_p$ which "does not move too much the points". This way the final approximating function is

$$g = p \circ h$$

Observe that

$$Dg(x) = Dp(h(x)) \circ Dh(x) : E \to F \text{ is a surjective operator.}$$

Problem: Given a Banach space E and a closed subset $X \subset E$, we look for diffeomorphisms between E and $E \setminus X$?

Theorem (Beginning of negligibility theory)

1. *(Bessaga, 1966) There exists a diffeomorphism $h : \ell_2 \to \ell_2 \setminus \{0\}$ so that $h = id$ outside the unit ball.*
2. *(West, 1969) For every compact set $K \subset \ell_2$ there exists a diffeomorphism $h : \ell_2 \to \ell_2 \setminus K$ so that h is as close to the identity as we want.*

THANK YOU FOR YOUR ATTENTION !!