Isometric Jordan isomorphisms of group algebras

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New Perspectives in Banach Spaces and Banach Lattices

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Contents

Problem: description of Jordan isomorphisms

Definition

A linear map $\Phi\colon A \to B$ between algebras is said to be...

- a homomorphism if $\Phi(ab) = \Phi(a)\Phi(b)\ \forall a, b \in A$,
- an anti-homomorphism if $\Phi(ab) = \Phi(b)\Phi(a)\ \forall a, b \in A$,
- a Jordan homomorphism if $\Phi(a^2) = \Phi(a)^2\ \forall a \in A$
  (eq., if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)\ \forall a, b \in A$).

Question

Is it clear than homomorphisms and anti-homomorphisms are Jordan homomorphisms. Is it possible to express every Jordan homomorphism using homomorphisms and/or anti-homomorphisms?

- Banach algebra: Banach space with a continuous product.
  Example: $C(K)$.
- C^*-algebra: Banach algebra with involution $(*)$.
  Example: $\mathbb{C}$, $z^* = \overline{z}$.
  Example: $C(K)$, $(f^*)(x) = \overline{f(x)}$.
  Example: $\mathcal{B}(H) =$ continuous operators on a Hilbert space.

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Previously known

(Kadison, 1951)

If A is a C^*-algebra with a predual (von Neumann algebra) and B is a C^*-algebra, then any isometric Jordan isomorphism from A onto B is the direct sum of an isometric isomorphism and an isometric anti-isomorphism.

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- $L^1(G) \subset M(G)$ via $g \mapsto \nu_g$, $\int_G f \, d\nu_g = \int_G fg \, d\mu$.
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- The group algebra is $L^1(G)$ with the convolution product.
- $L^1(G)$ is an ideal of $M(G)$ ($f * \nu, \nu * f \in L^1(G)$).
- We consider the strict topology in $M(G)$, which is given by the family of seminorms $(p_f)_{f \in L^1}$, $p_f(\nu) = \|f * \nu\|_1 + \|\nu * f\|_1$.

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- $L^1(G)$ is an ideal of $M(G)$ ($f * \nu, \nu * f \in L^1(G)$).
- We consider the strict topology in $M(G)$, which is given by the family of seminorms $(p_f)_{f \in L^1}$, $p_f(\nu) = \|f * \nu\|_1 + \|\nu * f\|_1$.
- $\overline{L^1(G)} = M(G) = \overline{\text{span}\{\delta_t : t \in G\}}$, where $\int_G f\, d\delta_t = f(t)$.

Theorem (Alaminos, Extremera, G., Villena (2024))

If $\Phi\colon L^1(G) \to L^1(H)$ is a contractive Jordan isomorphism, then one of the following holds:

- Φ *is an isometric isomorphism and it can be expressed as*

$$\Phi f(t) = c\chi(t) f(\varphi(t)) \quad (f \in L^1(t), t \in H)$$

where $\varphi\colon H \to G$ is an isomorphism, $\chi\colon H \to \mathbb{T}$ is a homomorphism and $c \in \mathbb{C}$.

- Φ *is and isometric anti-isomorphism and it can be expressed as*

$$\Phi f(t) = c\chi(t) f(\varphi(t)) \Delta_H(t^{-1}) \quad (f \in L^1(G), t \in H)$$

where $\varphi\colon H \to G$ is an anti-isomorphism, $\chi\colon H \to \mathbb{T}$ is a homomorphism and $c \in \mathbb{C}$.

Main idea of the proof

- $L^1(H) \subset M(H)$ as an ideal $(f * \nu, \nu * f \in L^1(H))$.
- The right translation by a measure $\nu \in M(H)$ is the map

$$R_\nu \colon L^1(H) \to L^1(H), \qquad R_\nu(f) = \nu * f \quad (f \in L^1(H)).$$

- (Wendel, 1952) If a linear map $T \colon L^1(H) \to L^1(H)$ is such that $T(f * g) = T(f) * g$ and $\|T(f)\| = \|f\|$ $\forall f, g \in L^1(H)$, then

$$\exists \lambda \in \mathbb{T},\ \theta \in H : T = \lambda R_{\delta_\theta}$$

- $\exists_1 \overline{\Phi} \colon M(G) \to M(H)$ Jordan isomorphism extending Φ.
- $\left\| \overline{\Phi}(\delta_t) * f \right\| = \|f\|$ $\forall f$, so $\exists \lambda(t) \in \mathbb{T}$, $\theta(t) \in H$:

$$R_{\overline{\Phi}(\delta_t)} = \lambda(t) R_{\theta(t)} \implies \overline{\Phi}(\delta_t) = \lambda(t) \delta_{\theta(t)}.$$

Application 1: two-sided zero product preservers

Definition

A linear map $\Phi\colon A \to B$ between algebras is said to be a two-sided zero product preserver if

$$ab = ba = 0 \implies \Phi(a)\Phi(b) = \Phi(b)\Phi(a) = 0.$$

Previously known

(Brešar, Godoy and Villena, 2022)

Continuous two-sided zero product preservers from $L^1(G)$ onto $L^1(H)$ are of the form $\Phi = \nu * \Psi$, where $\Psi\colon L^1(G) \to L^1(H)$ is a Jordan homomorphism and $\nu \in \mathcal{Z}(M(H))$ is invertible.

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Theorem (Alaminos, Extremera, G., Villena (2024))

Isometric two-sided zero product preservers from $L^1(G)$ onto $L^1(H)$ are of the form $\Phi = \alpha\delta_\theta * \Psi$, where $\alpha \in \mathbb{T}$, $\theta \in \mathcal{Z}(H)$, and $\Psi\colon L^1(G) \to L^1(H)$ is either an isometric isomorphism or an isometric anti-isomorphism.

Application 2: local isometric automorphisms

Definition

Let $\Phi\colon A \to B$ be a map. We say that...

- Φ satisfies locally a property if for each $a \in A$ there exists a map $\Phi_a\colon A \to B$ satisfying that property and such that $\Phi(a) = \Phi_a(a)$.
- Φ satisfies approximately locally a property if for each $a \in A$ there exists a sequence of maps $(\Phi_{a,n})$ satisfying that property and such that $\Phi(a) = \lim_n \Phi_{a,n}(a)$.

Question

Is every local or approximately local “something” actually a “something”?

Previously known

(Molnár and Zalar, 2000)

Under several hypothesis over G, local isometric automorphisms of $L^p(G)$, $1 \leq p \leq \infty$, are isometric automorphisms.

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Theorem (Alaminos, Extremera, G., Villena (2024))

Let $\Phi\colon L^1(G) \to L^1(G)$ *be a surjective bounded operator.*

- *If* Φ *is a local isometric automorphism and* G *is unimodular, then* Φ *is an isometric automorphism.*
- *If* Φ *is an approximately local isometric automorphism and* $G \in [MAP]$*, then* Φ *is an isometric automorphism.*

Previously known

(Molnár and Zalar, 2000)

Under several hypothesis over G, local isometric automorphisms of $L^p(G)$, $1 \leq p \leq \infty$, are isometric automorphisms.

Theorem (Alaminos, Extremera, G., Villena (2024))

Let $\Phi\colon L^1(G) \to L^1(G)$ be a surjective bounded operator.

- *If Φ is a local isometric automorphism and G is unimodular, then Φ is an isometric automorphism.*
- *If Φ is an approximately local isometric automorphism and $G \in [MAP]$, then Φ is an isometric automorphism.*

Steps of the proof

1. Φ is a Jordan isomorphism.
2. Φ is not an anti-isomorphism.

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