Projectional skeletons and Plichko in Lipschitz-free spaces

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New Perspectives in Banach spaces and Banach Lattices

Castro Urdiales, 11 July 2024

Research partially supported by grant PID2021-122126NB-C33 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU

Introduction

Lipschitz-Free spaces

Let M be a pointed metric space:

Definition: $\mathrm{Lip}_0(\mathrm{M})$

$$Lip_0(M) = \{f : M \to \mathbb{R} : \text{Lipschitz}, \ f(0) = 0\}$$

endowed with the norm $\|\cdot\|_L$.

Definition: Lipschitz-Free space

$$\mathfrak{F}(M) = \overline{\mathrm{span}}\{\delta(p) : p \in M\} = \overline{\mathrm{span}}\{m_{p,q} : (p,q) \in \widetilde{M}\}$$

Linearization

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \delta^M}\downarrow & & \downarrow{\scriptstyle \delta^N} \\ \mathfrak{F}(M) & \xrightarrow{\ \hat{f}\ } & \mathfrak{F}(N) \end{array}$$

Plichko property

Definition: r-Plichko, $r \geq 1$

X Banach is r-Plichko if exists $(\Delta, N) \subset X \times X^*$ with

- Δ is linearly dense in X.
- N is r-norming .
- for $f \in N$, $S_\Delta(f) = \{x \in \Delta \colon \langle f, x \rangle \neq 0\}$ is countable.

N is determined by Δ: X **Plichko, witnessed by Δ**.

Properties: X is Plichko then:

- X has the SCP.
- X is $\langle$LUR$\rangle$.
- X admits a strong M-basis.

Questions

Question A

Does there exists $\mathfrak{F}(M)$ non-Plichko?

Question B

Does there exists $\mathfrak{F}(M)$ without the SPC?

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Question $A\delta$

For wich M it holds that $\mathfrak{F}(M)$ is Plichko witnessed by **deltas**?

Question AM

For wich M it holds that $\mathfrak{F}(M)$ is Plichko witnessed by **molecules**?

Question Ax

For wich M it holds that $\mathfrak{F}(M)$ is Plichko witnessed by **x** ?

Plichko Property

Theorem: (Kubis09)

X is r-Plichko iff admits a **commutative r-projectional skeleton**.

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Definition: r-Projectional Skeleton (on X)

A **projectional skeleton** on X is $\{P_s\}_{s\in\Gamma}$ projections indexed by a directed, σ-complete poset Γ, with:

1. P_sX is separable for all $s \in \Gamma$.
2. $P_sP_t = P_tP_s = P_s$ whenever $s, t \in \Gamma$ and $s \leq t$.
3. If $(s_n)_n$ is an increasing sequence of indices in Γ, then $P_sX = \overline{\bigcup_{n\in\mathbb{N}} P_{s_n}X}$, for $s = \sup_{n\in\mathbb{N}} s_n$.
4. $X = \bigcup_{s\in\Gamma} P_sX$.

It is commutative if $P_sP_t = P_tP_s$ for all $s, t \in \Gamma$.
For $r \geq 1$, **r-projectional skeleton** whenever $\|P_s\| \leq r$.

Retractional Skeletons

Definition: *r-Retractional Skeleton (on M)*

A **Lipschitz retractional skeleton** on M is $\{R_s\}_{s\in\Gamma}$ Lipschitz retractions on M indexed by a directed, σ-complete poset Γ, with:

1. $R_s(M) \subset M$ separable, for all $s \in \Gamma$.
2. $R_s \circ R_t = R_t \circ R_s = R_s$, whenever $s, t \in \Gamma$ and $s \leq t$.
3. If $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence in Γ, then $R_s(M) = \overline{\bigcup_{n\in\mathbb{N}} R_{s_n}(M)}$ for $s = \sup_{n\in\mathbb{N}} s_n$.
4. $M = \bigcup_{s\in\Gamma} R_s(M)$.

It is commuative if $R_s \circ R_t = R_t \circ R_t$ for all $s, t \in \Gamma$.
For $r \geq 1$, ***r*-Lipschitz retractional skeleton** whenever $\|R_s\|_L \leq r$.

- By linearization: r-LRS on M implies r-PS in $\mathfrak{F}(M)$.
- $\mathfrak{F}(X)$ is Plichko whenever X is Plichko.
- Converse does not work.

Plichko **witnessed by deltas**

Characterización

Theorem (GMQ23)

M pointed metric, and $\lambda \geq 1$. TFAE:

1. For all $p \in M$ and for all $r < \frac{1}{\lambda}$, the ball $B(p, r \cdot d(p,0))$ is separable.

Proof:

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(1) $\Longleftrightarrow$ (5): By using Kalton characteriation of norming subspaces.

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4. M has a commut. λ-LRS $\{R_s\}_{s \in \Gamma}$ with $R_s(p) \in \{0, p\}$,$p \in M$.
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(3) $\Rightarrow$(4): Projections are induced by retractions.
(4) $\Rightarrow$(5): By McShane and composing with a suitable retraction.

Some examples and another characterization

$p \in [1,\infty]$, Γ uncountable: set $M_p = \bigcup_{\gamma\in\Gamma} E_\gamma$ where $E_\gamma = [0, e_\gamma] \subset \ell_p(\Gamma)$

M_p is a complete metric space of density character Γ. Satisfies (1).
So $\mathscr{F}(M_p)$ is 1-Plichko witnessed by deltas.

$N_2 = M_2 \cup \{e_\gamma + e_\nu : \gamma \neq \nu \in \Gamma\} \subset \ell_2(\Gamma)$

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Definition: separable λ-slab decomposition (M metric, $\lambda \geq 1$)

A family $\mathscr{S}$ of subsets of M with $M = \overline{\bigcup_{N\in\mathscr{S}} N}$, $N \setminus \{0\}$ open separable and for $p \in N$ there exists a countable subfamily $\mathscr{S}_p \subset \mathscr{S}$ such that $B(p, \lambda \cdot d(p,0)) \cap \bigcup_{N\in\mathscr{S}} N$ is contained in $B(p, \lambda \cdot d(p,0)) \cap \bigcup_{N\in\mathscr{S}_p} N$.

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Proposition (GMQ23) M metric, $\lambda \geq 1$. TFAE:

1. $\mathscr{F}(M)$ is λ-Plichko witnessed by a subset of $\delta(M)$.
2. M admits a separable λ-slab decomposition.

Plichko **witnessed by molecules**

Lipschitz-Free over $\mathbb{R}$-trees

Definition: $\mathbb{R}$-tree

A metric tree (T, d) is an $\mathbb{R}$-tree when for $x \neq y \in T$ there exists a unique arc $[x, y] \subset T$ (which) is isometric to the real line segment $[0, d(x, y)]$.

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Theorem (GMQ23)

$\mathfrak{F}(T)$ is Plichko witnessed by molecules as soon as T is an $\mathbb{R}$-tree.

Double Dandelion: Plichko witnessed by molecules but not by deltas.

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Idea of the proof:

- Pick $D \subset T$ dense such that $D \cap [0,p]$ is countable for every $[0,p]$.
- Family of separable subtrees of T. Indexed by a special tree that allows projecting points in non limite heights.
- $p \in D$, find $H(p)$ first separable subtree in the family of height $\alpha+1$.
- take $S(p)$ unique separable subtree of height α maximal inside $H(p)$.
- $\Delta = \{m_{p,P_{S(p)}(p)} \colon p \in D \text{ with } h(H(p)) = \alpha+1 \text{ for some } \alpha < \kappa\}$.

Extending the idea

Definition: λ-Lipschitz retractional tree on M

A family of λ-Lipschitz retractions $\{R_s\}_{s\in\Gamma}$ on M indexed by a rooted, σ-complete tree Γ such that height$(s) < \omega_1$ for all $s \in \Gamma$ such that:

1. $R_0(M) = \{0\}$ and $R_s(M)$ is separable for $s \in \Gamma$.
2. $R_s \circ R_t = R_t \circ R_s = R_{s\wedge t}$ for $s, t \in \Gamma$.
3. If $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence in Γ with $s = \sup_{n\in\mathbb{N}} s_n$, then $R_s(M) = \overline{\bigcup_{n\in\mathbb{N}} R_{s_n}(M)}$.
4. $M = \bigcup_{n\in\mathbb{N}} R_s(M)$.
5. For every σ-complete countable set A of Γ, the set $R_A(M)$ is closed and the retraction R_A is λ-Lipschitz.

Theorem GMQ23

$\mathfrak{F}(M)$ is Plichko witnessed by molecules as soon as M admits a LRT.

Uncountable eye: Plichko witnessed by molecules but not $\mathbb{R}$-tree.

Open Problems

Find a characterization of those metric spaces with $\mathfrak{F}(M)$ Plichko witnessed by molecules

Can they be characterized by the existence of some kind of rectractional skeleton?

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Dilucidate if there exists a non-Plichko Lipschitz-Free space.

Dilucidate if there exists a non-SCP Lipschitz-Free space.

Thank you all for your attention!