

M-ideals of compact operators and norm attaining operators

Manwook Han

Chungbuk National University

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Bases of this talk

S. K. Kim and M. Han

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Main goals :

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- Introduce WMP, CPP, ACPP and the basic theory of M-ideals
- Present three related questions
- Provide answers to two of them

Notations

- X , Y and J : real or complex Banach spaces,
- X^* : the topological dual of X ,
- If $J \subset X$, J^\perp : an annihilator of J in X^*
- B_X : the closed unit ball of X ,
- $\mathcal{L}(X, Y)$: the space of bounded linear operators from X to Y ,
- $\mathcal{K}(X, Y)$: the space of compact operators from X to Y ,
- For $T \in \mathcal{L}(X, Y)$, T^* : an adjoint operator for T
- $\|T\|_e = \text{dist}(T, \mathcal{K}(X, Y))$: The essential norm of T .

Introduction to the basic theory of M-ideals

M-ideal

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- For $1 < p \leq q < \infty$, $\mathcal{K}(\ell_p, \ell_q)$ is an M-ideal in $\mathcal{L}(\ell_p, \ell_q)$, But it is proper.
- $\mathcal{K}(X, c_0)$ is an M-ideal in $\mathcal{L}(X, c_0)$ for every Banach space X .

M-ideals of compact operators and ACPP

For the convenience, we will refer to the pair (X, Y) has the **M-ideal property** (MIP, in shorts) if $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$.

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Theorem (73' V. Zizler)

If the pair (X, Y) has the MIP

- (a) *(X, Y) satisfies the property that for $T \in \mathcal{L}(X, Y)$, if T^* does not attain its norm, then $\|T\| = \|T\|_e$.*

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MIP \Rightarrow ACPP

Compact perturbation property

We have named ACPP, deriving it from another property called the **Compact Perturbation Property**.

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Remark

- If the pair (X, Y) has the CPP, then X is reflexive
- If (X, Y) has the CPP, then (X, Y) has the ACPP.
- If X is reflexive and (X, Y) has the ACPP then (X, Y) has the CPP.

Weak maximizing property

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A pair (X, Y) is said to have the weak maximizing property (WMP, in shorts) if for every operator $T \in \mathcal{L}(X, Y)$, if there is non weakly null **maximizing sequence** for T , then T attains it's norm.

Definition (Maximizing sequence)

For an operator $T \in \mathcal{L}(X, Y)$, a sequence (x_n) in B_X is said to maximize T if $\|Tx_n\| \rightarrow \|T\|$. In this case, we says that (x_n) is a maximizing sequence for T .

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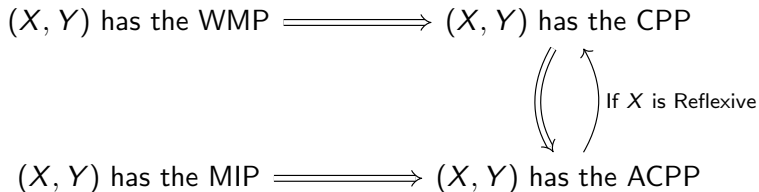
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Theorem (R. M. Aron, D. García and D. Pellegrino, E. V. Teixeira)

If a pair (X, Y) has the WMP, then (X, Y) has the CPP

Relations between properties



Question 1: Does ACPP implies the MIP?

First question

Example (Å. Lima, 79')

The pair (ℓ_1, ℓ_1) does not have the MIP.

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Theorem (M. Han and S. K. Kim)

The pair (ℓ_1, ℓ_1) does not have the ACPP.

Counter-example

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$\Rightarrow (E_0, E_0)$ does not have the MIP If E_0 is the Enflo-Davie subspace of c_0 .

I-polyhedrality

Definition (I-polyhedrality)

X has I-polyhedrality if it satisfies

$$(ExtB_{X^*})' \subset \{0\},$$

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Theorem (V. P. Fonf and L. Veselý, 04')

Suppose Γ is an index set with $\text{card}\Gamma = \text{dens}X$ where $\text{dens}X$ is the smallest cardinality of a dense subset of X .

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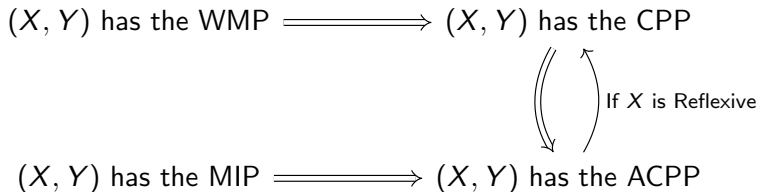
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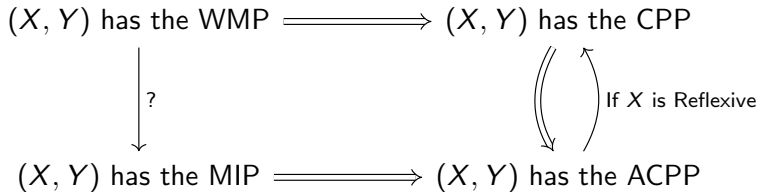
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Question 2: Does WMP implies the MIP?

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(N. J. Kalton and D. Werner) Suppose X admits **shrinking compact approximation of identity** (K_α) satisfying

$$\limsup_{\alpha} \|id_X - 2K_\alpha\| = 1.$$

Then the pair (X, Y) has the MIP iff the pair (X, Y) has **property (M)**

Property (M) and the Opial property

Definition

- ① (X, Y) has property (M) if for any elements $x \in X$ and $y \in Y$ with $\|y\| \leq \|x\|$, any contraction $T \in \mathcal{L}(X, Y)$ and any weakly null sequence $(x_n)_n$ in X , we have:

$$\limsup_n \|y + Tx_n\| \leq \limsup_n \|x + x_n\|$$

- ② (X, Y) has the Opial property if for any nonzero element $x \in X$, any contraction $T \in \mathcal{L}(X, Y)$ and any weakly null sequence $(x_n)_n$ in X , we have:

$$\limsup_n \|Tx_n\| < \limsup_n \|x + x_n\|$$

Property (M) and the Opial property

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- $\Rightarrow (E_q, E_q)$ has the WMP, but does not have the MIP.

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Recall our finding

Theorem (M. Han and S. K. Kim)

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For $2 \leq p < \infty$ and $1 < q \leq 2$, the following pairs have the WMP.

- ① $(\ell_p, L_p[0, 1])$
- ② $(\ell_p, c_p[0, 1])$
- ③ (ℓ_q, UT_q)

Where c_p is a Schatten class and UT_q is the space of upper triangle matrices in c_q .

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Question

Suppose the pair (X, Y) has the WMP.

- *Does (X, Y) possess property (M) ?*
- *Does (X, Y) possess the Opial property?*

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Price : An infinite amount of SOJU to enjoy with dinner

¡Muchas gracias!