M-ideals of compact operators and norm attaining operators

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Bases of this talk

S. K. Kim and M. Han
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Introduce WMP, CPP, ACPP and the basic theory of M-ideals
Present three related questions
Provide answers to two of them

Notations

- X, Y and J: real or complex Banach spaces,
- X^*: the topological dual of X,
- If $J \subset X$, $J^\perp$: an annihilator of J in X^*
- B_X: the closed unit ball of X,
- $\mathcal{L}(X, Y)$: the space of bounded linear operators from X to Y,
- $\mathcal{K}(X, Y)$: the space of compact operators from X to Y,
- For $T \in \mathcal{L}(X, Y)$, T^*: an adjoint operator for T
- $\|T\|_e = dist\,(T, \mathcal{K}(X, Y))$: The essential norm of T.

Introduction to the basic theory of M-ideals

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- For $1 < p \le q < \infty$, $\mathcal{K}(\ell_p, \ell_q)$ is an M-ideal in $\mathcal{L}(\ell_p, \ell_q)$, But it is proper.
- $\mathcal{K}(X, c_0)$ is an M-ideal in $\mathcal{L}(X, c_0)$ for every Banach space X.

M-ideals of compact operators and ACPP

For the convenience, we will refer to the pair (X, Y) has the **M-ideal property** (MIP, in shorts) if $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{L}(X, Y)$.

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If the pair (X, Y) has the MIP

(a) (X, Y) *satisfies the property that for* $T \in \mathcal{L}(X, Y)$, *if* T^* *does not attain its norm, then* $\|T\| = \|T\|_e$.

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MIP⇒ACPP

Compact perturbation property

We have named ACPP, deriving it from another property called the **Compact Perturbation Property**.

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Remark

- *If the pair* (X, Y) *has the CPP, then* X *is reflexive*
- *If* (X, Y) *has the CPP, then* (X, Y) *has the ACPP.*
- *If* X *is reflexive and* (X, Y) *has the ACPP then* (X, Y) *has the CPP.*

Weak maximizing property

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A pair (X, Y) is said to have the weak maximizing property(WMP, in shorts) if for every operator $T \in \mathcal{L}(X, Y)$, if there is non weakly null maximizing sequence for T, then T attains it's norm.

Definition (Maximizing sequence)

For an operator $T \in \mathcal{L}(X, Y)$, a sequence (x_n) in B_X is said to maximize T if $\|Tx_n\| \to \|T\|$. In this case, we says that (x_n) is a maximizing sequence for T.

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Theorem (R. M. Aron, D. García and D. Pellegrino, E. V. Teixeira)

If a pair (X, Y) has the WMP, then (X, Y) has the CPP

Relations between properties

(X, Y) has the WMP $\Longrightarrow$ (X, Y) has the CPP

(X, Y) has the CPP $\Downarrow$ (X, Y) has the ACPP

(X, Y) has the ACPP $\uparrow$ (X, Y) has the CPP [If X is Reflexive]

(X, Y) has the MIP $\Longrightarrow$ (X, Y) has the ACPP

Question 1: Does ACPP implies the MIP?

First question

Example (Å. Lima, 79')

The pair (ℓ_1, ℓ_1) *does not have the MIP.*

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The answer is NO

Theorem (M. Han and S. K. Kim)

The pair (ℓ_1, ℓ_1) does not have the ACPP.

Counter-example

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$\Rightarrow$ (E_0, E_0) does not have the MIP If E_0 is the Enflo-Davie subspace of c_0.

I-polyhedrality

Definition (I-polyhedrality)

X has I-polyhedrality if it satisfies

$$(ExtB_{X^*})' \subset \{0\},$$

where $(ExtB_{X^*})'$ *is the set of* w^**-cluster points of* $ExtB_{X^*}$.

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Theorem (V. P. Fonf and L. Veselý, 04')

Suppose Γ *is an index set with* $card\Gamma = densX$ *where* $densX$ *is the smallest cardinality of a dense subset of* X.
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Question 2: Does WMP implies the MIP?

Property (M)

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(N. J. Kalton and D. Werner) Suppose X admits **shrinking compact approximation of identity** (K_α) satisfying

$$\limsup_{\alpha} \|id_X - 2K_\alpha\| = 1.$$

Then the pair (X, Y) has the MIP iff the pair (X, Y) has property (M)

Property (M) and the Opial property

Definition

1. (X, Y) *has property (M) if for any elements* $x \in X$ *and* $y \in Y$ *with* $\|y\| \leq \|x\|$, *any contraction* $T \in \mathcal{L}(X, Y)$ *and any weakly null sequence* $(x_n)_n$ *in* X, *we have:*

$$\limsup_n \|y + Tx_n\| \leq \limsup_n \|x + x_n\|$$

2. (X, Y) *has the Opial property if for any nonzero element* $x \in X$, *any contraction* $T \in \mathcal{L}(X, Y)$ *and any weakly null sequence* $(x_n)_n$ *in* X, *we have:*

$$\limsup_n \|Tx_n\| < \limsup_n \|x + x_n\|$$

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$\Rightarrow$ (E_q, E_q) has the WMP, but does not have the MIP.

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Recall our finding

Theorem (M. Han and S. K. Kim)

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For $2 \leq p < \infty$ and $1 < q \leq 2$, the following pairs have the WMP.

1. $(\ell_p, L_p[0,1])$
2. $(\ell_p, c_p[0,1])$
3. (ℓ_q, UT_q)

Where c_p is a Schatten class and UT_q is the space of upper triangle matrices in c_q.

Open questions

Question

Suppose the pair (X, Y) has the WMP.

- *Does (X, Y) possese property (M)?*
- *Does (X, Y) possese the Opial property?*

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Price : An infinite amount of SOJU to enjoy with dinner

¡Muchas gracias!