Strong Subdifferentiable points in Lipschitz-free spaces

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Let $(X, \|\cdot\|)$ be a Banach space and $x \in S_X$.

Recall that $\|\cdot\|$ is Gâteaux differentiable at x if

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t} \quad \text{exists for every } h \in X.$$

If the above limit exists uniformly for $h \in S_X$, then $\|\cdot\|$ is Fréchet differentiable at x.

Šmulyan lemma :

(i) $\|\cdot\|$ is G-differentiable at $x \in S_X$ $\iff$ whenever $f_n, g_n \in S_{X^*}$ satisfy $f_n(x) \to 1$ and $g_n(x) \to 1$, then $f_n - g_n \xrightarrow{w^*} 0$.

(ii) $\|\cdot\|$ is F-differentiable at $x \in S_X$ $\iff$ whenever $f_n, g_n \in S_{X^*}$ satisfy $f_n(x) \to 1$ and $g_n(x) \to 1$, then $\|f_n - g_n\| \to 0$.

As a consequence, we have :

(i) $\|\cdot\|$ is G-differentiable at $x \in S_X \iff \exists$ unique $f \in S_{X^*}$: $f(x) = 1$, i.e., x exposes B_{X^*} at $f \in S_{X^*}$. ($\iff f$ is w^*-exposed by x)

(ii) $\|\cdot\|$ is F-differentiable at $x \in S_X \iff \exists$ unique $f \in S_{X^*}$: x strongly exposes B_{X^*} at $f \in S_{X^*}$. ($\iff f$ is w^*-strongly exposed by x).

Let $x \in S_X$ be given. Consider $D(x) := \{ x^* \in X^* : x^*(x) = \|x^*\| = 1 \}$ which is nonempty, convex and w^*-compact.

Put $\tau(x,h) := \lim\limits_{t \to 0^+} \dfrac{\|x + th\| - \|x\|}{t}$

Note that $t \longmapsto \dfrac{\|x + th\| - \|x\|}{t}$ is an increasing function of t and bounded below.

$$\tau(x,h) := \lim_{t \to 0^+} \frac{\|x+th\| - 1}{t}$$

If $x^* \in D(x)$, i.e., $x^*(x) = \|x^*\| = 1$, then

$$x^*(h) = \frac{x^*(x+th) - x^*(x)}{t} \leqslant \frac{\|x+th\| - 1}{t}$$ which implies that

$$\max\{ x^*(h) : x^* \in D(x) \} \leqslant \tau(x,h).$$

In fact, we can observe that $\max\{ x^*(h) : x^* \in D(x) \} = \tau(x,h)$.

(The idea is to consider $f_0 : \mathrm{span}\{x, h\} \longrightarrow \mathbb{R}$ given by $f_0(ax+bh) = a + b\,\tau(x,h)$.

Check that f_0 is a well-defined linear form with norm one.

Extend it to $f : X \longrightarrow \mathbb{R}$ by Hahn-Banach. Then $f \in D(x)$ and $f(h) = \tau(x,h)$.)

It is usually said that the norm on X is subdifferentiable.

If the equality

$$\tau(x,h) = \lim_{t \to 0+} \frac{\|x + th\| - 1}{t} = \max\{x^*(h) : x^* \in D(x)\}$$

holds uniformly for $h \in B_X$, we say that $\|\cdot\|$ is strongly subdifferentiable at $x \in S_X$.

Having in mind that the set $\{x^*(h) : x^* \in D(x)\}$ turns to be a singleton when $\|\cdot\|$ is Gâteaux differentiable at x, observe that

$$\text{Gâteaux differentiability} + \text{SSD} \iff \text{Fréchet differentiability.}$$

(For this reason, SSD is called a 'non-smooth extension of Fréchet differentiability'.)

Note that SSD is more general than F-differentiability since every finite dimensional normed space is SSD. (thanks to Dini's theorem)

Recall that

(i) $\|\cdot\|$ is G-differentiable at $x \in S_X$ $\iff$ $\exists$ unique $f \in S_{X^*}$: $f(x) = 1$, i.e., x exposes B_{X^*} at $f \in S_{X^*}$. ($\iff$ f is w^*-exposed by x)

(ii) $\|\cdot\|$ is F-differentiable at $x \in S_X$ $\iff$ $\exists$ unique $f \in S_{X^*}$: x strongly exposes B_{X^*} at $f \in S_{X^*}$. ($\iff$ f is w^*-strongly exposed by x).

Šmulyan type lemma for SSD (Franchetti, Payá 1993)

$D(x) = \{ x^* \in S_{X^*} : x^*(x) = 1 \}$

$\|\cdot\|$ is SSD at $x \in S_X$ $\iff$ $D(x)$ is strongly exposed by x, i.e., whenever $(x_n^*) \subseteq B_X$ satisfies that $x_n^*(x) \longrightarrow 1$, then $\mathrm{dist}(x_n^*, D(x)) \longrightarrow 0$.

Examples

- The sup-norm on c_0 is SSD.
- the sup-norm on ℓ_∞ is SSD at $x = (x_n)$ $\iff$ $\|x\| \notin \{ |x_n| : |x_n| < \|x\|_\infty \}'$ derived set
- the canonical norm on ℓ_1 is SSD at x $\iff$ x is finitely supported.

From the examples, we can see that SSD concept is really different from Fréchet differentiability.

To see some what SSD and Fréchet have in common ...

- $\|\cdot\|^*$ is F-differentiable $\Longrightarrow$ X is reflexive

  $\|\cdot\|^*$ is SSD $\Longrightarrow$ X is reflexive

- X admits F-diff norm $\Longrightarrow$ X is Asplund

  X admits SSD norm $\Longrightarrow$ X is Asplund

  (Godefroy, 1993)

From now on, we dive into the Lipschitz-free world

$$\mathcal{F}(M) = \overline{\text{span}}\ \delta(M) \subseteq \text{Lip}_0(M)^*$$

Recall $m_{x,y} := \dfrac{\delta(x) - \delta(y)}{d(x,y)}$ for $x \neq y$ in M is called an (elementary) molecule.

Theorem (Becerra, López, Rueda Zoca, 2018) Let M be an infinite pointed metric space. If M is unbounded or not uniformly discrete, then the norm on $\mathcal{F}(M)$ is octahedral.

$\Longrightarrow$ Notice that if $\|\cdot\|$ on X is octahedral, then for any $x \in S_X$, $\exists\ (f_n), (g_n) \subseteq B_{X^*}$ such that $f_n(x) \longrightarrow 1$, $g_n(x) \longrightarrow 1$ and $\|f_n - g_n\| \longrightarrow 2$. (i.e., "2-rough")

Of course, any $x \in S_X$ cannot be Fréchet differentiable point.

To have a Fréchet differentiable point in $\mathcal{F}(M)$, the underlying M must be bounded and uniformly discrete.

Perhaps, the first result in this line :

Theorem (Procházka, Rueda Zoca, 2018) Let M be bounded uniformly discrete.

Let $x_1, \cdots, x_n \in M \setminus \{0\}$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Put $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i, 0}$. Then TFAE :

(1) μ is a Fréchet differentiable point in $\mathcal{F}(M)$

(2) μ is a Gâteaux differentiable point in $\mathcal{F}(M)$

(3) $\forall z \in M$, there is $i \in \mathbb{N}$ such that $d(x_i, z) + d(z, 0) = d(x_i, 0)$.

(i.e., M is the union of the segments $[0, x_i]$)

This result was extended by Aliaga and Rueda Zoca in 2020.

The key result is a characterization of sum of convex series of molecules via some geometric condition called "cyclical monotonicity".

Theorem (Aliaga, Rueda Zoca, 2020) Let M be bounded uniformly discrete.

Let $\mu \in \operatorname{span} S(M)$ be of norm one and write $\mu = \sum_{i=1}^{n} \lambda_i m_{x_i, y_i}$ for some $x_i \neq y_i$ in M and $\lambda_i \in \mathbb{R}^+$ with $\sum_{i=1}^{n} \lambda_i = 1$. Let $f \in S_{Lip_0(M)}$ such that $\langle f, \mu \rangle = 1$.

Then TFAE:

(1) μ is a Fréchet differentiable point in $\mathcal{F}(M)$

(2) μ is a Gâteaux differentiable point in $\mathcal{F}(M)$

(3) for every $j \neq k$ in $\{1, \cdots, n\}$, there is $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}$ that contains j and k and such that

$$d(x_{i_1}, y_{i_1}) + d(x_{i_2}, y_{i_2}) + \cdots + d(x_{i_m}, y_{i_m}) = d(x_{i_1}, y_{i_2}) + d(x_{i_2}, y_{i_3}) + \cdots + d(x_{i_m}, y_{i_1})$$

and

for every $z \in M$, there are $s \neq t$ in $\{x_1, y_1, \cdots, x_n, y_n\}$ such that $z \in [s, t]$ and $f(t) - f(s) = d(t, s)$.

Recall that if $\tau(x,h) = \lim_{t \to 0+} \frac{\|x+th\| - 1}{t} = \max\{x^*(h) : x^* \in D(x)\}$

holds uniformly for $h \in B_X$, we say that $\|\cdot\|$ is strongly subdifferentiable at $x \in S_X$.

Moreover,

$\|\cdot\|$ is SSD at $x \in S_X$ $\iff$ $D(x)$ is strongly exposed by x, i.e., whenever $(x_n^*) \subseteq B_{X^*}$

satisfies that $x_n^*(x) \longrightarrow 1$, then $\operatorname{dist}(x_n^*, D(x)) \longrightarrow 0$.

Q) Can $\mathcal{F}(M)$ be SSD ? (i.e., the norm on $\mathcal{F}(M)$ is SSD at every point)

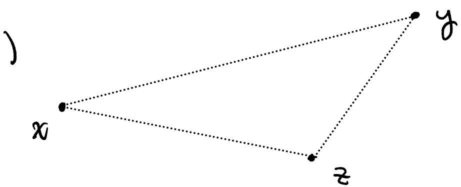
A) when M is infinite, then NO ! By Cuth-Doucha-Wojtaszczyk (2017), $\mathcal{F}(M)$ contains

an complemented subspace which is isomorphic to ℓ_1.

thus, $\mathcal{F}(M)$ is SSD $\iff$ M is finite.

To investigate SSD points in $\mathcal{F}(M)$, we start with molecules $m_{x,y}$ in $\mathcal{F}(M)$.

Recall the Gromov product of x, y with respect to z $(x, y, z \in M)$



$$G_z(x,y) = d(x,z) + d(z,y) - d(x,y)$$

Remark Gromov product is a useful tool to study some extremal structures of $B_{\mathcal{F}(M)}$.

(see Ivakno, Kadets, Werner (2007), Garcia-Lirola, Procházka, Rueda (2018), and Aliaga, Pernecka (2020))

Let say (x,y) has (G) if $\exists\ \eta = \eta(x,y) > 0$ such that $G_z(x,y) > \eta \quad \forall z \in M \setminus \{x,y\}$.

Why we consider (G) ?

Lemma (x,y) has (G) $\Longleftrightarrow$ $\exists f \in S_{Lip_0(M)}$: $f(m_{x,y}) = 1$ and

$$\sup\{ |f(m_{p,q})| : (p,q) \neq (x,y) \} < r < 1$$

Proposition (Cobollo, Dantas, Hájek, J. 2024+) Let M be any metric space, $x \neq y \in M$.

(x,y) has (G) $\Longrightarrow$ $m_{x,y}$ is an SSD point in $\mathcal{F}(M)$.

(sketch of the proof)

Take $f \in S_{\mathrm{Lip}_0(M)}$ so that $f(m_{x,y}) = 1$ and $\sup\{ |f(m_{p,q})| : (p,q) \neq (x,y) \} < r < 1$.

Let $\varepsilon > 0$ and $g \in S_{\mathrm{Lip}_0(M)}$ so that $g(m_{x,y}) > 1 - r_\varepsilon$, where $0 < r_\varepsilon < \frac{\varepsilon}{4-\varepsilon}(1-r)$.

Enough to find $\varphi \in D(m_{x,y})$ so that $\|\varphi - g\| \sim 0$.

Put $h := (1-\frac{\varepsilon}{4}) g + \frac{\varepsilon}{4} \cdot f$. Observe that

$$h(m_{x,y}) = (1-\frac{\varepsilon}{4}) g(m_{x,y}) + \frac{\varepsilon}{4} f(m_{x,y}) > (1-\frac{\varepsilon}{4})(1-r_\varepsilon) + \frac{\varepsilon}{4}.$$

For $(p,q) \neq (x,y)$, we have

$$|h(m_{p,q})| \leq (1-\frac{\varepsilon}{4}) |g(m_{p,q})| + \frac{\varepsilon}{4} |f(m_{p,q})| \leq (1-\frac{\varepsilon}{4}) + \frac{\varepsilon}{4} \cdot r < h(m_{x,y})$$

$\Longrightarrow$ $\|h\| = h(m_{x,y}) \sim 1$.

Consequently, $h/\|h\| \in D(m_{x,y})$ and $\left\| h/\|h\| - g \right\| \sim 0$. $\square$

On way to construct a metric in such a way that every pair of points has (G):

Let (M, d) be given and $r > 0$.

Consider $d_r : M \times M \longrightarrow \mathbb{R}$ given by $d_r(x, y) = \begin{cases} d(x,y) + r & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Then with this metric d_r, we have $G_z(x, y) \geq r \quad \forall z \in M \setminus \{x, y\}$.

Thus, every molecule in $\mathcal{F}(M, d_r)$ is an SSD point.

In fact, in this case ...

Theorem A (Cobollo, Dantas, Hájek, J. 2024+) Let (M, d) be any metric space, $r > 0$.

Every finitely supported element in $\mathcal{F}(M, d_r)$ is an SSD point.

Actually we are going to prove the following :

Theorem B (Cobollo, Dantas, Hájek, J. 2024+) Let (M, d) be any metric space, $r > 0$.

Every element of

$$\left\{ \sum_{i=1}^{n} \lambda_i \frac{\delta(x_i) - \delta(y_i)}{d(x_i, y_i) + r} : \begin{array}{l} \exists f \in S_{Lip_0(M,d)} \text{ such that } f(x_i) - f(y_i) = d(x_i, y_i) \ \forall i = 1, \dots, n. \\ \lambda_i \in \mathbb{R}^+, \ i = 1, \dots, n \text{ with } \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \end{array} \right\}$$

is an SSD point in $\mathcal{F}(M, d_r)$.

(Proof of Theorem A) Let us put $\tilde{d} := d_r$ and let $\mu \in \mathcal{F}(M, \tilde{d}_r)$ be finitely supported.

Note that $\tilde{d}_r = (d_r)_r = d_{2r}$. Write $\mu = \sum_{i=1}^{n} \lambda_i \frac{\delta(x_i) - \delta(y_i)}{\tilde{d}(x_i, y_i) + r}$ optimal representation.

Find $f \in S_{Lip_0(M, \tilde{d}_r)}$ so that $f(\mu) = 1$. i.e., $f(x_i) - f(y_i) = \tilde{d}(x_i, y_i) + r \ \forall i = 1, \dots, n$.

Note that $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are disjoint. If not, assume for instance $x_1 = y_2$.

Then $f(x_2) = f(y_2) + \tilde{d}(x_2, y_2) + r = f(x_1) + \tilde{d}(x_2, x_1) + r = f(y_1) + \tilde{d}(x_1, y_1) + \tilde{d}(x_2, x_1) + 2r$

$$\geq f(y_1) + \tilde{d}(y_1, x_2) + 2r.$$

Put $N = \{x_1, y_1, \cdots, x_n, y_n\}$. WLOG, $0 \in \{y_1, \cdots, y_n\}$.

Define $\tilde{f} : N \to \mathbb{R}$ by $\tilde{f}(x_i) := f(x_i) - r$ and $\tilde{f}(y_i) := f(y_i)$ for $i = 1, \cdots, n$.

Note that $\tilde{f}(x_i) - \tilde{f}(y_i) = \tilde{d}(x_i, y_i)$ for every $i = 1, \cdots, n$.

Summarizing, every finitely supported element in $\mathcal{F}(M, \tilde{d}_r) = \mathcal{F}(M, d_{2r})$ is an SSD point.

<u>Claim</u> $\tilde{f} \in S_{\mathrm{Lip}_0(N, \tilde{d})}$.

This is true thanks to that $\tilde{d}$ is already distorted.

For instance if $\tilde{f}(x_i) < \tilde{f}(y_j)$, then

$$
\begin{aligned}
|\tilde{f}(x_i) - \tilde{f}(y_j)| = \tilde{f}(y_j) - \tilde{f}(x_i) &= \tilde{f}(y_j) - \tilde{f}(x_j) + \tilde{f}(x_j) - \tilde{f}(x_i) \\
&= -\tilde{d}(y_j, x_j) + f(x_j) - f(x_i) \\
&\leq -\tilde{d}(y_j, x_j) + \tilde{d}_r(x_j, x_i) \\
&= -\tilde{d}(y_j, x_j) + \tilde{d}(x_j, x_i) + r \overset{!}{\leq} \tilde{d}(x_i, y_j)
\end{aligned}
$$

Extend $\tilde{f}$ to a norm one element in $\mathcal{F}(M, \tilde{d})$.

Apply Theorem A to $(M, \tilde{d})$ with $\mu \in \mathcal{F}(M, \tilde{d}_r)$ and $\tilde{f} \in S_{\mathrm{Lip}_0(M, \tilde{d})}$.

Theorem B (Cobollo, Dantas, Hájek, J. 2024+) Let (M,d) be any metric space, $r > 0$.

Every element of

$$\left\{ \sum_{i=1}^{n} \lambda_i \frac{\delta(x_i) - \delta(y_i)}{d(x_i,y_i) + r} : \begin{array}{l} \exists f \in S_{Lip_0(M,d)} \text{ such that } f(x_i) - f(y_i) = d(x_i,y_i) \ \forall i = 1, \cdots, n. \\ \lambda_i \in \mathbb{R}^+,\ i = 1, \cdots, n \text{ with } \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \end{array} \right\}$$

is an SSD point in $\mathcal{F}(M, d_r)$.

(Sketch of the proof).

Let μ be an element of the above set with $f \in S_{Lip_0(M,d)}$.

Note that $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ are disjoint and let $N := \{x_1, y_1, \cdots, x_n, y_n\}$.

WLOG, $0 \in \{y_1, \cdots, y_n\}$. Note that $f|_N \in S_{Lip_0(N,d)}$, with $\|f|_N\|_\infty \leq \mathrm{diam}(N)$.

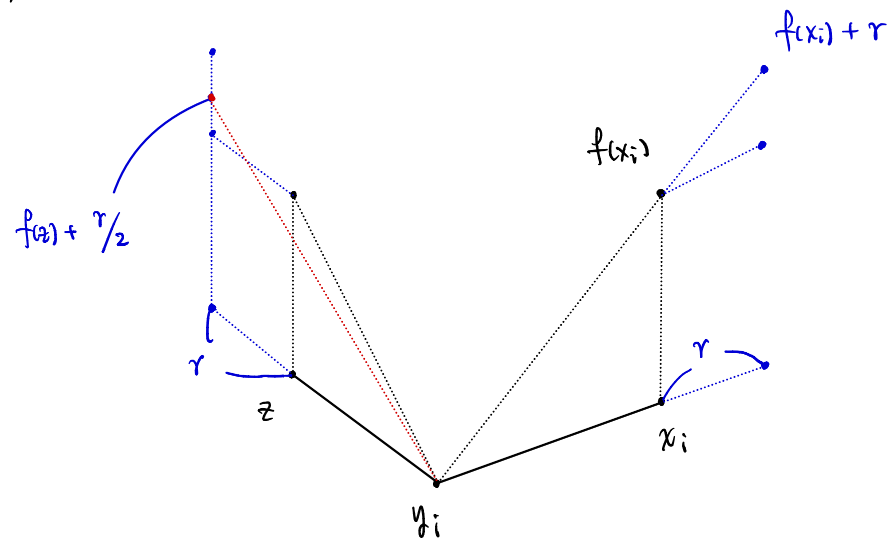
Apply McShane to get a norm one extension of $f|_N$ to (M,d) with sup-norm $\leq \mathrm{diam}(N)$.

We still denote this by f.

<u>Claim</u> : $\exists \varphi \in S_{Lip_0(M,d_r)}$: $\sup\{ |\varphi(m_{p,q})| : p \notin N \text{ or } q \notin N \} < \beta < 1$ & $\varphi(\mu) = 1$

Next we define $f_r \in \mathrm{Lip}_0(M, d_r)$ by

$$f_r(z) = \begin{cases} f(x_i) + r & \text{if } z = x_i, \ i = 1, \cdots, n. \\ f(y_i) & \text{if } z = y_i, \ i = 1, \cdots, n. \\ f(z) + r/2 & \text{if } z \notin N. \end{cases}$$



Note that for $i, j \in \{1, \cdots, n\}$,

(1) $|f_r(x_i) - f_r(y_j)| \leq d(x_i, y_j) + r$

(2) $|f_r(x_i) - f_r(z)| \leq d(x_i, z) + r/2$

(3) $|f_r(y_j) - f_r(z)| \leq d(y_j, z) + r/2$.

Suppose that M is bounded for a moment.

$$\Longrightarrow \quad |f_r(m_{x_i, z})| \leq \frac{d(x_i, z) + r/2}{d(x_i, z) + r} \leq \frac{k + r/2}{k + r} \quad \text{where } k = \mathrm{diam}(M).$$

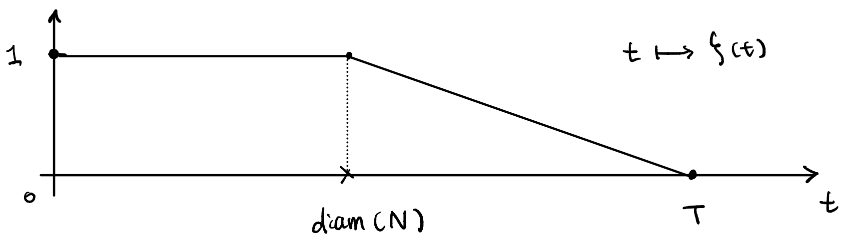
Similarly, $|f_r(m_{z,y_i})| \le \dfrac{k + r/2}{k + r}$ for every $z \notin N$.

Observe that $f_r(x_i) - f_r(y_i) = f(x_i) + r - f(y_i) = d(x_i, y_i) + r$; hence

$$f_r(\mu) = f_r\left(\sum_{i=1}^{n} \lambda_i \frac{\delta(x_i) - \delta(y_i)}{d(x_i, y_i) + r}\right) = \sum_{i=1}^{n} \lambda_i \frac{f_r(x_i) - f_r(y_i)}{d(x_i, y_i) + r} = \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and}$$

$$\sup\{ |f_r(m_{p,q})| : p \notin N \text{ or } q \notin N \} \le \frac{k + r/2}{k + r}.$$

In fact, the boundedness of M can be dropped by considering $\xi : \mathbb{R}^+ \cup \{0\} \longrightarrow [0,1]$



and replacing f_r by $z \longmapsto f_r(z)\,\xi(d(0,z))$. So, we can still have

$$f_r(\mu) = 1 \quad \text{and} \quad \sup\{ |f_r(m_{p,q})| : p \notin N \text{ or } q \notin N \} \le \frac{k + r/2}{k + r} \quad \text{for some } k > 0.$$

Now, we are close to the end of the proof. Let $\varepsilon > 0$ and $g \in S_{\mathrm{Lip}_0(M, d_r)}$ so that

$g(\mu) > 1-\rho$ where $\rho = \rho(\varepsilon, \mu) > 0$ small enough so that

$$(I) := (1-\sqrt{\varepsilon}) + \sqrt{\varepsilon}\cdot\left(\frac{k + r/2}{k+r}\right) + \frac{\mathrm{diam}(N)\cdot\varepsilon}{r} < (1-\sqrt{\varepsilon})(1-\rho) + \sqrt{\varepsilon}$$

Put $h := (1-\sqrt{\varepsilon})\, g + \sqrt{\varepsilon}\cdot f_r$. (then $\|h - g\| \leq \sqrt{\varepsilon}$)

Then $h(\mu) > (1-\sqrt{\varepsilon})(1-\rho) + \sqrt{\varepsilon} > (I) > (1-\sqrt{\varepsilon}) + \sqrt{\varepsilon}\cdot\left(\frac{k + r/2}{k+r}\right)$

$$\geq \sup\{\, |h(m_{p,q})| : p \notin N \text{ or } q \notin N \,\}.$$

this shows that $\|h\| = \|h|_N\|$. Note that $\mathcal{F}(N, d_r)$ is SSD (due to its dimension)

Find $\varphi \in \mathrm{Lip}_0(N, d_r)$ so that $\varphi(\mu) = \|\varphi\| = \|h\|$ and $\|\varphi - h|_N\| \sim 0$.

Finally, define $\psi : (M, d_r) \longrightarrow \mathbb{R}$ by
$$\psi(z) = \begin{cases} \varphi(z) & \forall\, z \in N \\ h(z) & \forall\, z \in M \setminus N. \end{cases}$$

Then $\|\psi - h\| \sim 0$; hence $\|\psi - g\| \sim 0$. Note also that $\psi(\mu) = \varphi(\mu) = \|h\|$.

It is enough to show that $\|\psi\| = \|h\|$. A nontrivial case : $p \in N$ and $q \in M \setminus N$.

Then
$$|\psi(m_{p,q})| = \frac{|\varphi(p) - h(q)|}{d(p,q) + r}$$
$$\leq \frac{|\varphi(p) - h(p)|}{d(p,q) + r} + |h(m_{p,q})|$$
$$\leq \frac{\|\varphi - h|_N\| \cdot \operatorname{diam}(N)}{r} + (1-\sqrt{\varepsilon}) + \sqrt{\varepsilon} \cdot \left(\frac{k + r/2}{k+r}\right)$$
$$\leq (I)$$
$$< (1-\sqrt{\varepsilon})(1-\rho) + \sqrt{\varepsilon} < h(\mu).$$

$\square$

Thank you for your attention!

References

[1] R.J. Aliaga and A. Rueda Zoca, *Points of differentiability of the norm in Lipschitz-free spaces*. J. Math. Anal. Appl. 489 (2020), no.2, 124171, 17 pp.

[2] R.J. Aliaga and E. Pernecká, *Supports and extreme points in Lipschitz-free spaces*. Rev. Mat. Iberoam. 36 (2020), no. 7, pp. 2073–2089.

[3] J.B. Guerrero, G. López Pérez and A. Rueda Zoca, *Octahedrality in Lipschitz-free Banach spaces*, Proc. Roy. Soc. Edinburgh Sect. A. 148, **3** (2018), 447–460

[4] C. Cobollo, S. Dantas, P. Hájek and M. Jung, *On the strongly subdifferentiable points in Lipschitz-free spaces*, preprint, https://arxiv.org/abs/2406.01269

[5] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*. Arch. Math. **46** (1986), pp. 76–84.

[6] C. Franchetti and R. Payá, *Banach spaces with strongly subdifferentiable norm*. Boll. Uni. Mat. Ital. VII-B (1993), pp. 45–70.

[7] L.C. García-Lirola, A. Procházka and A. Rueda Zoca, *A characterisation of the Daugavet property in spaces of Lipschitz functions*, J. Math. Anal. Appl. **464**, 1 (2018), 473–492

[8] L.C. García Lirola, C. Petitjean, A. Procházka and A. Rueda Zoca, *Extremal Structure and Duality of Lipschitz Free Spaces*. Mediterr. J. Math. (2018) 15: 69.

[9] Y. Ivakhno, V. Kadets and D. Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. 101 (2007) 261–279.

[10] A. Procházka and A. Rueda Zoca, *A characterisation of octahedrality in Lipschitz-free spaces*, Ann. Inst. Fourier (Grenoble) 68 (2018), no. 2, 569–588.