Separating all diameter two properties in spaces of Lipschitz functions

Jaan Kristjan Kaasik

University of Tartu, Estonia

Joint work with Andre Ostrak and Rainis Haller

We consider only nontrivial real Banach spaces.

Given a Banach space X we denote

- the closed unit ball of X by B_X;
- the unit sphere of X by S_X;
- the dual space of X by X^*.

A **slice** of the unit ball B_X of a Banach space X is a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Definition

X has the

- **LD2P** if every slice of B_X has diameter 2;
- **D2P** if every nonempty relatively weakly open subset of B_X has diameter 2;
- **SD2P** if every convex combination of slices of B_X has diameter 2;
- **SSD2P** if, for every finite family $\{S_1, \ldots, S_n\}$ of slices of B_X and every $\varepsilon > 0$, there exist $x_1 \in S_1, \ldots, x_n \in S_n$, and $y \in B_X$ with $\|y\| > 1 - \varepsilon$ such that $x_i \pm y \in S_i$ for every $i \in \{1, \ldots, n\}$.

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The following implications hold for a general X:

$$\text{SSD2P} \Longrightarrow \text{SD2P} \Longrightarrow \text{D2P} \Longrightarrow \text{LD2P}$$

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The following implications hold for a general X^*:

$$\begin{array}{ccccccc} \text{SSD2P} & \Longrightarrow & \text{SD2P} & \Longrightarrow & \text{D2P} & \Longrightarrow & \text{LD2P} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{w*-SSD2P} & \Longrightarrow & \text{w*-SD2P} & \Longrightarrow & \text{w*-D2P} & \Longrightarrow & \text{w*-LD2P} \end{array}$$

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- The **space of Lipschitz functions** $\mathrm{Lip}_0(M)$ is defined as

$$\mathrm{Lip}_0(M) = \{f : M \to \mathbb{R} \colon f \text{ is Lipschitz and } f(0) = 0\},$$

with the norm

$$\|f\| = \sup\left\{\frac{f(x) - f(y)}{d(x, y)} : x, y \in M, x \neq y\right\}.$$

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- The **Lipschitz-free space** $\mathcal{F}(M)$ is defined as

$$\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta_m : m \in M\} \subset \mathrm{Lip}_0(M)^*,$$

where $\delta_m \in \mathrm{Lip}_0(M)^*$, $m \in M$, is defined by

$$\delta_m(f) = f(m), \qquad f \in \mathrm{Lip}_0(M).$$

- $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$.

Let M be a complete metric space.

Theorem (Avilés, Martínez-Cervantes, 2019)

The following are equivalent:

(i) *M is a length space;*

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(vi) *$\mathcal{F}(M)$ has the LD2P.*

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(vi) $\mathcal{F}(M)$ *has the LD2P.*

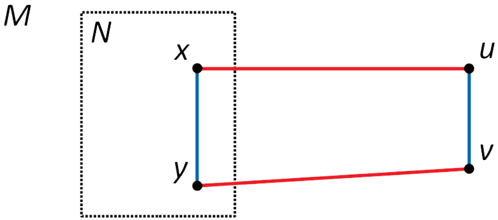
Theorem (K, Veeorg, 2023)

The space $\mathcal{F}(M)$ *never has the SSD2P.*

Definition (Procházka, Rueda Zoca, 2018)

M has the **LTP** if, for every $\varepsilon > 0$ and every finite $N \subset M$, there exist $u, v \in M$, $u \neq v$, satisfying, for all $x, y \in N$,

$$(1-\varepsilon)\big(d(x,y)+d(u,v)\big) \leq d(x,u)+d(y,v).$$



Theorem (Procházka, Rueda Zoca, 2018)

M has the LTP if and only if $\mathrm{Lip}_0(M)$ has the w^*-SD2P.

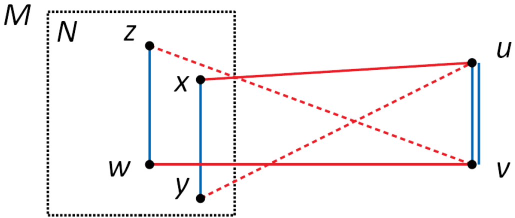
Definition (Ostrak, 2020)

M has the **SLTP** if for, for every $\varepsilon > 0$ and every finite $N \subset M$, there exist $u, v \in M$, $u \neq v$, satisfying, for all $x, y \in N$,

$$(1-\varepsilon)\big(d(u,v)+d(x,y)\big) \leq d(x,u)+d(y,v)$$

and, for all $x, y, z, w \in N$,

$$(1-\varepsilon)\big(2d(u,v)+d(x,y)+d(z,w)\big) \leq d(x,u)+d(y,u)+d(z,v)+d(w,v).$$



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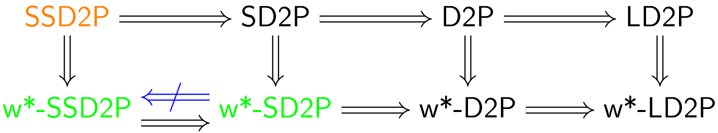
- M has the LTP $\iff$ $\mathrm{Lip}_0(M)$ w^*-SD2P
- M has the SLTP $\iff$ $\mathrm{Lip}_0(M)$ w^*-SSD2P
- $\mathrm{Lip}_0(M)$ has the SSD2P (J. Langemets, A. Rueda Zoca, 2020)
    - if M is unbounded
    - if M is not uniformly discrete;
    - if $M = K_n$, $n \neq 2$.



- Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Given a Banach space X and a sequence $x_n \in B_X$, define $F\colon X^* \to \mathbb{R}$ by

$$F(f) = \lim_{\mathcal{U}} f(x_n), \quad \text{for all } f \in X^*.$$

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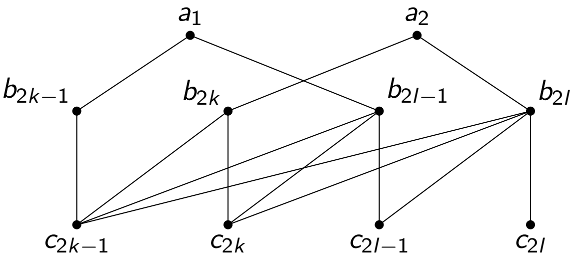
  Then $F \in B_{X^{**}}$.
- If $f \in S(F, \alpha)$ for some $\alpha > 0$, then there are infinitely many $n \in \mathbb{N}$ so that $f(x_n) > 1 - \alpha$.

Example (Haller, K, Ostrak, 2024)

Let $M = \{a_1, a_2\} \cup \{b_k, c_k \colon k \in \mathbb{N}\}$ be the metric space where, for every $k, l \in \mathbb{N}$, $k \leq l$,

$$d(a_1, b_{2k-1}) = d(a_2, b_{2k}) = d(c_k, b_l) = 1$$

and the distance between the elements is 2 in all other cases.



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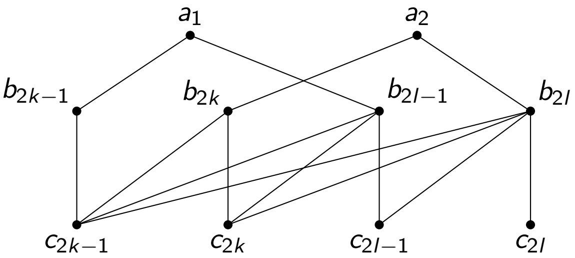
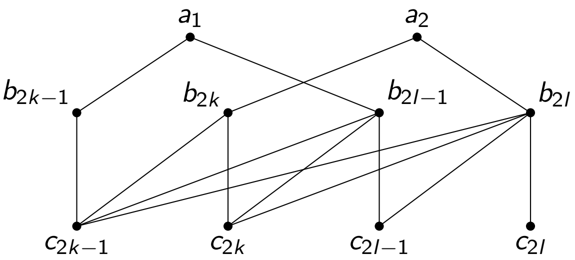


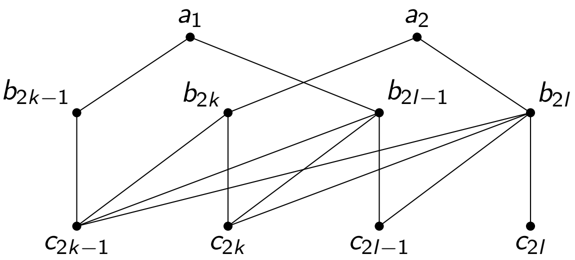
SLTP: Let $N \subset M$ be finite. Choose $u = c_k, v = b_k$ so that $c_l, b_l \notin N$ for all $l \geq k$. Then $d(u, v) = 1$ and $d(u, N) = 2$.



No LD2P: Let $0 = c_1$, define $F \in B_{\mathrm{Lip}_0(M)^*}$ by

$$F\colon f \longmapsto \lim_{\mathcal{U}} f\left(m_{b_{2n-1},b_{2n}}\right),$$

and let $G = \frac{1}{2}\left(m_{a_1,a_2} + F\right)$. Note that $G \in S_{\mathrm{Lip}_0(M)^*}$.



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and let $G = \frac{1}{2}\left(m_{a_1,a_2} + F\right)$. Note that $G \in S_{\mathrm{Lip}_0(M)^*}$.
Let $f \in S(G, \alpha)$. For any $k, l \in \mathbb{N}$, $k \leq 2l - 1$,

$$f(b_{2l-1}) - f(b_{2l}) \approx 2 \implies f(c_k) \approx (\max f + \min f)/2;$$

$$f(a_1) \approx \max f \implies f(b_{2k-1}) \geq \max f - 1;$$

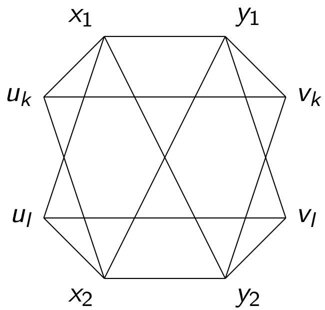
$$f(a_2) \approx \min f \implies f(b_{2k}) \leq \min f + 1.$$

Example (Ostrak, 2020)

Let $M = \{x_1, x_2, y_1, y_2\} \cup \{u_i, v_i : i \in \mathbb{N}\}$ be the metric space where, for all $i, j \in \{1, 2\}$, $k \in \mathbb{N}$,

$$d(x_i, u_k) = d(y_i, v_k) = d(x_i, y_j) = d(u_k, v_k) = 1$$

and the distance between two elements is 2 in all other cases. Then M has the LTP but lacks the SLTP.



Example (Haller, Ostrak, Põldvere, 2022)

Let $M = \{a_i, u^i_m, v^i_m \colon i \in \{1,2,3\}, m \in \mathbb{N}\}$ be the metric space where, for all $i, j \in \{1,2,3\}$, $i \neq j$, $m \in \mathbb{N}$,

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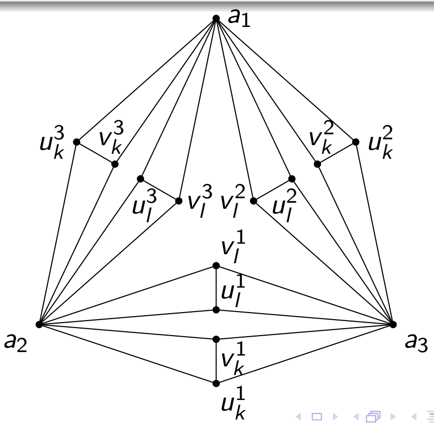
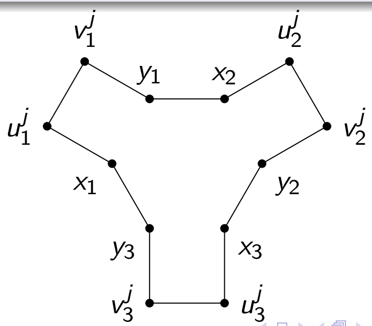


Example (New result)

Let $M = \{x_i, y_i, u_i^j, v_i^j \mid i \in \{1,2,3\}, j \in \mathbb{N}\}$ be a metric space where for every $i \in \{1,2,3\}$ and $j \in \mathbb{N}$,

$$\begin{aligned} d(x_i, u_i^j) &= d(u_i^j, v_i^j) = d(v_i^j, y_i) \\ &= d(x_2, y_1) = d(x_3, y_2) = d(x_1, y_3) = 1 \end{aligned}$$

and otherwise the distance is 2.



Open questions

- Is there a Banach space X with the SSD2P such that X^{**} lacks the SSD2P?

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- Find metric characterisations for all diameter two properties in spaces of Lipschitz functions.

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