Weak limits of consecutive projections onto convex sets 1.

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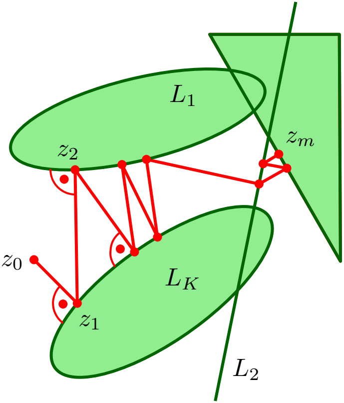
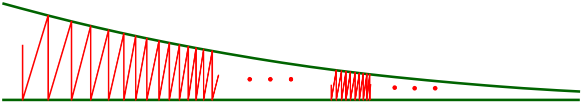
DO THE ITERATES OF PROJECTIONS CONVERGE?



K fixed, *e.g.* $K = 5$
$C_1, C_2, \dots, C_K \subset \mathbb{R}^d$ or ℓ_2
closed convex sets

$k_1, k_2, \dots \in \{1, 2, \dots, K\}$ be arbitrary
$z_n = P_{k_n} z_{n-1}$ sequence of projections

Do the iterates converge to a point in the intersection $A = \bigcap C_k$,
if it is non-empty?



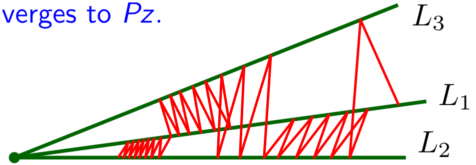
Linear subspaces: (non)-convergence well understood

$L_1, L_2, \dots, L_K$ closed subspaces of a Hilbert space H
$z_n = P_{k_n} z_{n-1}$ iterates of orthonormal projections of a point z
Pz = orthogonal projection of z onto $L_1 \cap L_2 \cap \cdots \cap L_K$

If $H = \mathbb{R}^d$, then $\{z_n\}$ converges to Pz.
[$d = 2$ von Neumann '49, Práger '60]

If $H = \ell_2$, then $\{z_n\}$ converges to Pz *weakly*. [Amemiya, Ando '65]
$(\sum_1^N z_n)/N$ converges in norm for almost all $\{k_n\} \in \{1, \dots, K\}^{\mathbb{N}}$.

If the iterates are (quasi)cyclic *e.g.* $P_1 P_2 P_3 P_1 P_2 P_3 \dots$,
[Halperin '62], [Sakai '95]
or if in every step we project onto the most distant subspace,
[Borodin, Kopecká '20]
then $\{z_n\}$ converges to Pz.



MOST but NOT ALL random projections converge

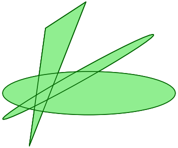
Let H be an infinite dimensional Hilbert space.

For all closed subspaces $L_1, \dots, L_K$ of H, and $z_0 \in H$ the sequence of iterates defined by $z_n = P_{k_n} z_{n-1}$ converges in norm for almost all choices of the sequence of indices $\{k_n\} \in \{1, \dots, K\}^{\mathbb{N}}$.
almost all $=$ full measure in the product measure on $\{1, \dots, K\}^{\mathbb{N}}$
[Lira Melo, da Cruz Neto, Machado de Brito '22]
almost all $=$ dense-G_δ in the product topology on $\{1, \dots, K\}^{\mathbb{N}}$
[Daylen Thimm '24]

There exist 3 closed subspaces L_1, L_2, L_3 with the following property. For every $0 \neq z_0 \in H$ there is a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $z_n = P_{k_n} z_{n-1}$ does not converge in norm.
[Eva Kopecká,Vladimír Müller, Adam Paszkiewicz, '14,'17]

If $L_1^\perp + \dots + L_K^\perp$ is NOT closed in H, then a counterexample as above can be constructed within these subspaces. [Kopecká 20]

Convex sets



H Hilbert space
closed and convex $C_1, C_2, \ldots, C_K \subset H$
$\bigcap C_i \neq \emptyset$

$z_n = P_{k_n} z_{n-1}$ iterates of the nearest point projections of a point z

If $H = \mathbb{R}^d$ then $\{z_n\}$ converges. [Dye, Kuczumow, Lin, Reich '96]

There exist 2 closed and convex sets $C, D \subset \ell_2$ with $0 \in C \cap D$, and a sequence $\{z_n\}$ of iterates of nearest point projections on these sets which does NOT converge in norm. [Hundal '04]

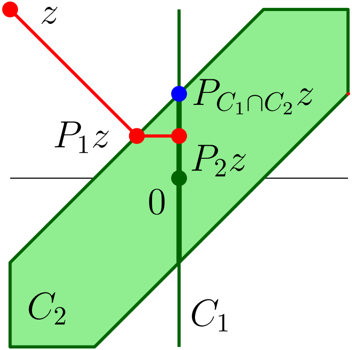
If $K = 3$ then $\{z_n\}$ converges *weakly*. [Bruck '82], [Dye, Reich '92]
For $K = 4$ this is not known!

If the iterates are (quasi)cyclic, or if in every step we project onto the most distant set, then $\{z_n\}$ converges weakly. [Bregman '65]

If C_i's are symmetric then $\{z_n\}$ converges *weakly*; [Dye, Reich '92] if we, moreover, project onto the most distant sets then $\{z_n\}$ converges *in norm*. [Borodin, Kopecká 23]

Remotest projections

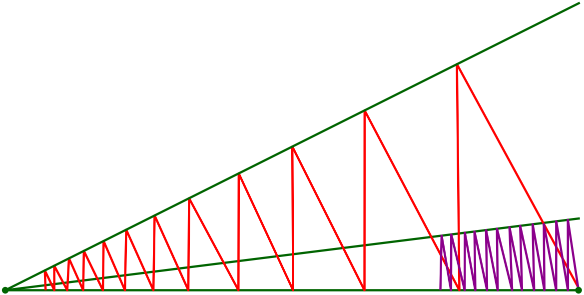
H Hilbert space, closed and convex $C_1, C_2, \ldots, C_K \subset H$,
$C = \bigcap C_i \neq \emptyset$,
$z_n = P_{k_n} z_{n-1}$ iterates of the nearest point projections of a point z



If C_i's are symmetric and if we, moreover, in every step project onto the most distant set, then $\{z_n\}$ converges *in norm*.
The limit can be different from $P_C(z)$.

remotest projections onto symmetric convex sets converge

Let all C_α, $\alpha \in \Omega$, be closed, convex, and symmetric subsets of a Hilbert space. (Hence $0 \in \bigcap C_\alpha$.) Then the sequence of remotest projections $z_n = P_{\alpha_n} z_{n-1}$ where
$\mathrm{dist}\,(z_n, C_{\alpha_n}) = \max_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$
converges in norm for any starting element $z_0 \in H$.
[Borodin, Kopecká '23]



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[Borodin, Kopecká '23]

relaxing symmetry by uniform quasi-symmetry:
$\forall \alpha : x \in C_\alpha \Rightarrow -\frac{1}{10} x \in C_\alpha$
Here we WLOG assume $0 \in C_\alpha \subset B(0,5)$ for all α;
$\frac{1}{10}$ can be replaced by a $\delta \in [0,1]$.

relaxing remoteness: choose α_n so that
$\operatorname{dist}(z_n, C_{\alpha_n}) \geq \frac{1}{2} \max_{\alpha \in \Omega} \operatorname{dist}(z_n, C_\alpha)$

More generally, take $t_n \in [0,1]$ instead of $\frac{1}{2}$
satisfying the condition
$\forall (a_i) \in \ell_2$ with $a_i \geq 0 :\ \liminf_{m \to \infty} \frac{a_m}{t_m} \sum_{i=1}^{m} a_i = 0$
invented by [Temlyakov '02] for all C_α hyperplanes; best possible.

remote projections onto SYMMETRIC convex sets converge in NORM

relaxing remoteness: choose α_n so that
$\mathrm{dist}\,(z_n, C_{\alpha_n}) \geq t_n \max_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$

where $t_n \in [0,1]$ satisfy the condition

(T) $\forall (a_i) \in \ell_2$ with $a_i \geq 0$: $\liminf_{m\to\infty} \frac{a_m}{t_m} \sum_{i=1}^m a_i = 0$

Remark: $\sum t_n/n = \infty \;\Rightarrow$ (T) $\Rightarrow \; \sum t_n^2 = \infty$

Theorem

For a sequence $\{t_n\}_{n=0}^\infty \subset [0,1]$, the following two statements are equivalent:

(i) *The sequence $\{t_n\}$ satisfies the condition (T).*

(ii) *For any family $\{C_\alpha\}_{\alpha\in\Omega}$ of closed and convex sets in a Hilbert space H which is uniformly quasi-symmetric and for any starting element $z_0 \in H$ the sequence $\{z_n\}$ of remote projections converges in norm to a point in $\bigcap C_\alpha$.*

remote projections onto convex sets converge weakly

relaxing remoteness: choose α_n so that
$\mathrm{dist}\,(z_n, C_{\alpha_n}) \geq t_n \max_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$

where $t_n \in [0, 1]$ satisfy the following condition

there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \ldots, t_{n+K}$ is greater than δ.

Theorem

Assume $\{C_\alpha\}$ is a family of closed and convex sets in a Hilbert space H with a nonempty intersection $C = \bigcap_{\alpha\in\Omega} C_\alpha$. Let the sequence $\{t_n\}$ in $[0, 1]$ satisfy the following condition: there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \ldots, t_{n+K}$ is greater than δ. Then the sequence of remote projections converges weakly to some point of C for any starting element $x_0 \in H$.

Come to visit.