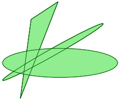
Weak limits of consecutive projections onto convex sets 2.

Eva Kopecká

University of Innsbruck
Austria

Finitely many convex sets



H Hilbert space

closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$

$z_n = P_{k_n} z_{n-1}$ iterates of the nearest point projections

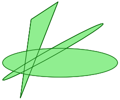
Does $\{z_n\}$ converge at least weakly?

If $H = \mathbb{R}^d$ then $\{z_n\}$ converges.
[Dye, Kuczumow, Lin, Reich '96]

There exist 2 closed and convex sets $C, D \subset \ell_2$ with $0 \in C \cap D$, and a sequence $\{z_n\}$ of iterates of nearest point projections on these sets which does NOT converge in norm.
[Hundal '04]

There exist 3 closed subspaces $L_1, L_2, L_3 \subset \ell_2$ with the following property. For every $0 \neq z_0 \in H$ there is a sequence $k_1, k_2, \dots \in \{1, 2, 3\}$ so that the sequence of iterates defined by $z_n = P_{k_n} z_{n-1}$ does not converge in norm.
[Eva Kopecká, Vladimír Müller, Adam Paszkiewicz, '14,'17]

Finitely many convex sets



H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$
Is it true that $W = \{0\}$?

W is non-empty and weakly compact.
If W contains an element of maximal norm, then $W = \{0\}$.

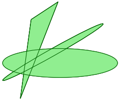
If C_i's are subspaces, or just symmetric, then $\{z_n\}$ converges *weakly*. [Amemyia and Ando '65], [Dye, Reich '92]

If $K = 3$ then $\{z_n\}$ converges *weakly*. For $K = 4$ this is not known!
[Bruck '82], [Dye, Reich '92]

If $\{k_n\}$ is periodic, then the iterates $\{z_n\}$ converge *weakly*.
[Bregman '65]

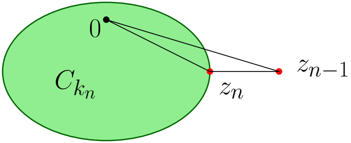
If we project onto the most distant sets then $\{z_n\}$ converges *weakly*. [Borodin, Kopecká 23]

Finitely many convex sets



H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$

projections 1-Lipschitz $\Rightarrow \{|z_n|\}$ decreasing $\Rightarrow W$ weakly compact



$$|z_{n-1} - z_n|^2 \le |z_{n-1}|^2 - |z_n|^2$$
$$\Rightarrow \; |z_n - z_{n+m}|^2 \le m(|z_n|^2 - |z_{n+m}|^2)$$

$$\Rightarrow \operatorname{dist}(z_n, C_{k_{n\pm m}}) \to 0 \; (n \to \infty) \text{ for any fixed } m.$$

Hence, if $\{k_n\}$ is periodic, then $w \in \bigcap_{i-1}^{K} C_i = \{0\}$ for every $w \in W$, hence $\{z_n\}$ converges weakly to 0.

For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \dots, K\}$ such that $w \in C_{J(w)}$. Here $C_J = \bigcap_{j \in J} C_j$.
Therefore $|J(w)| \ge 2$ for each $w \in W$.
If $w \ne 0$, then $|J(w)| < K$, since $\bigcap C_i = \{0\}$.

Finitely many convex sets

H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$
For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \dots, K\}$ such that $w \in C_{J(w)}$. Here $C_J = \cap_{j \in J} C_j$.

Theorem (Borodin, Kopecká '22)

For each $w \in W$, $w \neq 0$, one can find another element $w' \in W$ with the following properties:

- (i) $|J(w') \setminus J(w)| \geq 1$;
- (ii) $|J(w') \cap J(w)| \geq 2$
- (iii) $|J(w')| \geq 3$;
- (iv) $\langle w' - w, a \rangle \geq 0$ *for every* $a \in C_{J(w)}$.

In particular $|w'| > |w|$ *since by (iv) for a=w*
$|w'|^2 = |w|^2 + |w' - w|^2 + 2\langle w' - w, w \rangle \geq |w|^2 + |w - w'|^2$.

Finitely many convex sets

H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$
For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \dots, K\}$ such that $w \in C_{J(w)}$. Here $C_J = \bigcap_{j \in J} C_j$.

If $K = 2$ or $K = 3$ then $\{z_n\}$ converges *weakly*.
For $K = 4$ this is not known!

Proof.
If $K = 2$, then $\{k_n\} = \{1, 2, 1, 2, \dots\}$ is periodic,
hence $\{z_n\}$ converges *weakly*.
Assume that $K = 3$ and that there is $w \in W \setminus \{0\}$.
By Theorem there is $w' \in W$ with $|w'| > |w|$ and
$J(w') = \{1, 2, 3\}$. Hence $w' = 0$ which is a contradiction. □

Finitely many convex sets

H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$
For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \dots, K\}$ such that $w \in C_{J(w)}$. Here $C_J = \bigcap_{j \in J} C_j$.

Theorem (Borodin, Kopecká '22)

If $W \neq \{0\}$, then here exist $w \neq w' \in W$ so that

$$w = \mathrm{weak} \lim_{k\to\infty} z_{n_k}, \qquad w' = \mathrm{weak} \lim_{k\to\infty} z_{m_k},$$

where $n_1 < m_1 < n_2 < m_2 < \dots$, and $k_n \in J(w) \cap J(w')$ for any $n \in \bigcup_j (n_j, m_j)$. Consequently,

(i) $\langle w' - w, a \rangle \geq 0$ *for every* $a \in C_{J(w)}$,

(ii) $\langle w' - w, b \rangle \geq 0$ *for every* $b \in C_{J(w')}$.

Finitely many convex sets

H Hilbert space, $k_1, k_2, \dots \in \{1, 2, \dots, K\}$
closed and convex $C_1, C_2, \dots, C_K \subset H$, $\bigcap C_i = \{0\}$
$W = W(z_0)$ be all partial weak limits of the sequence
$z_n = P_{k_n} z_{n-1}$ of iterates of the nearest point projections of $z_0 \in H$
For $w \in W$, we denote by $J(w)$ the maximal subset of $\{1, \dots, K\}$ such that $w \in C_{J(w)}$. Here $C_J = \cap_{j \in J} C_j$.

Assume all the convex sets C_i are quasi-symmetric: if $a \in C_i$ then $-\lambda a \in C_i$ for some $\lambda = \lambda(a, i) > 0$. Then $W = \{0\}$.
In particular, if all C_i are closed linear subspaces of H then $\{z_n\}$ converges weakly.

Proof.

Assuming $W \neq \{0\}$ we take the two different elements $w, w' \in W$ from Theorem. Using (i) of Theorem for $a = w$ gives $\langle w' - w, w \rangle \geq 0$ and $|w'|^2 \geq |w|^2 + |w - w'|^2$. Using (ii) of Theorem for $b = -\lambda w'$ gives $\langle w' - w, -\lambda w' \rangle \geq 0$ and $|w|^2 \geq |w'|^2 + |w - w'|^2$. Hence $w = w'$, a contradiction. □