Weak limits of consecutive projections onto convex sets
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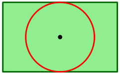
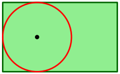
degrees of symmetry

bounded, closed and convex $C, C_\alpha \subset H$ Hilbert space

C is symmetric: $x \in C \Rightarrow -x \in C$

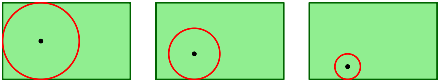


C is somewhat symmetric: $\forall x \in C\, \exists \delta_x \in (0,1] : -\delta_x x \in C$



C is quasi symmetric: $\exists \delta \in (0,1] : x \in C \Rightarrow -\delta x \in C$

C is somewhat symmetric $\Leftrightarrow$ C is quasi symmetric



The family C_α, $\alpha \in \Omega$, is uniformly quasi symmetric:

$\exists \delta \in (0,1]\, \forall \alpha : x \in C_\alpha \Rightarrow -\delta x \in C_\alpha$

somewhat symmetric $\Leftrightarrow$ almost symmetric

bounded, closed and convex $C \subset H$ Hilbert space

C is somewhat symmetric: $\forall x \in C \, \exists \delta_x \in (0,1] : \; -\delta_x x \in C$
C is quasi symmetric: $\exists \delta \in (0,1] : \; x \in C \Rightarrow -\delta x \in C$

C is somewhat symmetric $\Leftrightarrow$ C is quasi symmetric

$\Leftarrow$: clear

Proof 1 $\Rightarrow$: Suppose $\inf_{x \in C} \delta_x = 0$.
Choose $x_n \in C$ having $\delta_{x_n} < 1/3^n$.
Then $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \in C$, hence $\delta_x > 0$ implying after some computations $\delta_{x_k} > 1/3^k$ for large k's - a contradiction.

Proof 2 $\Rightarrow$: (Baire category thm)
Both sets $C \cap (-C)$ and $\operatorname{conv}(C \cup (-C))$ generate norms on $\operatorname{span} C$ in which $\operatorname{span} C$ is a Banach space. These norms are equivalent in view of the open mapping theorem.
Hence $\inf_{x \in C} \delta_x > 0$.

remote projections onto SYMMETRIC convex sets converge in NORM

relaxing remoteness: choose α_n so that
$\mathrm{dist}\,(z_n, C_{\alpha_n}) \geq t_n \sup_{\alpha\in\Omega} \mathrm{dist}\,(z_n, C_\alpha)$

where $t_n \in [0,1]$ satisfy the condition

(T) $\forall (a_i) \in \ell_2$ with $a_i \geq 0 :\ \liminf_{m\to\infty} \frac{a_m}{t_m} \sum_{i=1}^{m} a_i = 0$

Remark: $\sum t_n/n = \infty \ \Rightarrow (\mathrm{T}) \Rightarrow \ \sum t_n^2 = \infty$

Theorem

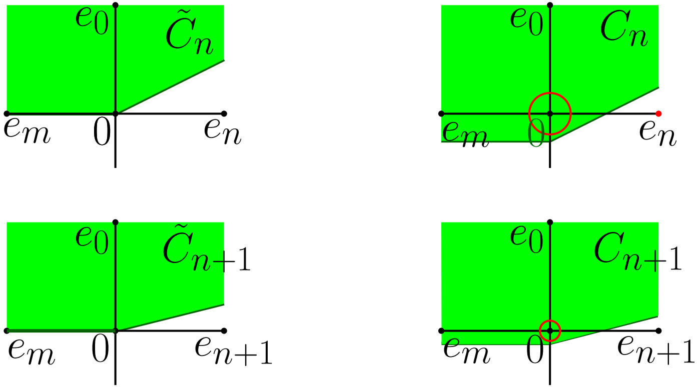
For a sequence $\{t_n\}_{n=0}^{\infty} \subset [0,1]$, the following two statements are equivalent:

(i) *The sequence $\{t_n\}$ satisfies condition (T).*

(ii) *For any family $\{C_\alpha\}_{\alpha\in\Omega}$ of closed and convex sets in a Hilbert space H which is uniformly quasi-symmetric and for any starting element $z_0 \in H$ the sequence $\{z_n\}$ of remote projections converges in norm to a point in $\bigcap C_\alpha$.*

UNIFORM almost symmetry needed

There are C_n, $n \in \mathbb{N}$, closed, convex, and (NOT uniformly!!) almost symmetric subsets of ℓ_2 and a sequence $\{z_n\}$ of remotest projections onto these sets which does NOT converge in norm.



convergence of remote projections
$\Rightarrow$ convergence of quasi-periodic projections

IF the remoteness parameters $t_n \in [0, 1]$ satisfy condition (T) THEN for any family $\{C_\alpha\}_{\alpha\in\Omega}$ of closed and convex sets in a Hilbert space H which is uniformly quasi-symmetric the sequence $\{z_n\}$ of remote projections converges in norm to a point in $\bigcap C_\alpha$.

Corollary

Assume $C_1, \ldots, C_K$ are finitely many closed, convex and quasi-symmetric subsets of H with a nonempty intersection $C = \bigcap_1^K C_j$. Assume $\{\alpha(n)\}$ is a quasi-periodic sequence of the indices $1, \ldots, K$. Then the sequence $x_{n+1} = P_{\alpha(n)} x_n$ of nearest point projections converges in norm to a point in C.

Proof.

Define $b_n = \max_k \operatorname{dist}(x_n, C_k)$ and $t_n = |z_{n+1} - z_n|/b_n$.
$\Rightarrow$ each interval I of length K contains $n \in I$ with $t_n \geq 1/(6K)$ (if in each of the K steps we project onto a set that is too close we couldn't have visited them all)
$\Rightarrow \sum t_n/n = \infty \Rightarrow$ (T) is satisfied.

all C_α's contain the same ball $\Rightarrow$ any product of projections converges in norm

Theorem

Let each closed convex set C_α contain the ball $B(0, r)$, $r > 0$.

(a) *The sequence of remote projections converges in norm for each starting element $x_0 \in H$ and for any sequence $\{t_n\}$. In particular, random projections converge.*

(b) *If, moreover, $\sum t_n^2 = \infty$, then the limit point w belongs to $\bigcap_{\alpha\in\Omega} C_\alpha$, and the rate of convergence is estimated by*

$$|z_n - w| \leq 2|z_0| \prod_{k=0}^{n-1} \left(1 - \frac{t_k^2 r^2}{|z_0|^2}\right)^{1/2}.$$

remote projections onto convex sets converge weakly

relaxing remoteness: choose α_n so that
$\operatorname{dist}(z_n, C_{\alpha_n}) \geq t_n \sup_{\alpha\in\Omega} \operatorname{dist}(z_n, C_\alpha)$

where $t_n \in [0, 1]$ satisfy the following condition

there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \dots, t_{n+K}$ is greater than δ.

Theorem

Assume $\{C_\alpha\}$ is a family of closed and convex sets in a Hilbert space H with a nonempty intersection $C = \bigcap_{\alpha\in\Omega} C_\alpha$. Let the sequence $\{t_n\}$ in $[0, 1]$ satisfy the following condition: there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \dots, t_{n+K}$ is greater than δ. Then the sequence of remote projections converges weakly to some point of C for any starting element $x_0 \in H$.

$\liminf_{n\to\infty} t_n > 0$ not enough for weak convergence

Example

Let H be an infinite dimensional Hilbert space. Then there exists a countable family of closed convex sets in H with non-empty intersection and a sequence of remote projections on this family which does not converge weakly and its weakness parameters satisfy $\liminf_{n\to\infty} t_n > 0$.

weak convergence of remote projections
$\Rightarrow$ weak convergence of quasi-periodic projections

IF there are $\delta > 0$ and $K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ at least one of the values $t_n, \ldots, t_{n+K}$ is greater than δ,
THEN the sequence of remote projections converges weakly

Corollary

Assume $C_1, \ldots, C_K$ are finitely many closed and convex subsets of H with a nonempty intersection $C = \bigcap_1^K C_j$. Assume $\{\alpha(n)\}$ is a quasi-periodic sequence of the indices $1, \ldots, K$. Then the sequence $x_{n+1} = P_{\alpha(n)} x_n$ of nearest point projections converges weakly to a point in C for any starting point $x_0 \in H$.

Proof.

Define $b_n = \max_k \operatorname{dist}(x_n, C_k)$ and $t_n = |z_{n+1} - z_n|/b_n$.
$\Rightarrow$ each interval I of length K contains $n \in I$ with $t_n \geq 1/(6K)$
(if in each of the K steps we project onto a set that is too close we couldn't have visited them all)
$\Rightarrow$ weak convergence.

all C_α's contain the same brick $\Rightarrow$ any product of projections converges weakly

Theorem

Let $\{C_\alpha\}$ be a family of closed convex subsets of a Hilbert space H. Assume $0 \in C = \bigcap_\alpha C_\alpha$ and that $\bigcup_{n\in\mathbb{N}} nC$ is dense in H. Then the sequence of remote projections converges weakly for any sequence $\{t_n\}$ of weakness parameters. In particular, random projections converge weakly in this case.

Proof.

projections 1-Lipschitz $\Rightarrow$ the sequence $\{|z_n - v|\}$ is decreasing for every $v \in C$, hence has a limit. Since

$$|z_n - v|^2 = |z_n|^2 - 2\langle z_n, v\rangle + |v|^2,$$

the sequence of scalar products $\{\langle z_n, v\rangle\}$ has a limit for every $v \in C$ as well. $\Rightarrow$ the sequence of iterates $\{z_n\}$ converges weakly $\square$

the results above appear in the paper

P.A. Borodin, E. Kopecká, *Convergence of remote projections onto convex sets*, Pure Appl. Funct. Anal. 8 (2023), 1603-1620.