On Ramsey-type properties of the distance in nonseparable spheres - III

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Proof.

$$\mathcal{X} = \overline{span}(\{\chi_S : S \in \mathcal{S}\} \cup c_0(\mathbb{R})\} \subseteq \ell_\infty(\mathbb{R}),$$

$\mathcal{S}$ is a family of some sequences in $\mathbb{R}$ converging to different limits. □

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Assume that that there is a coloring $c : [\omega_1]^2 \to \{0, 1\}$ *such that for every pairwise disjoint family* $\mathcal{F}$ *of finite subsets of* ω_1 *and* $i \in \{0, 1\}$ *there are* $A, B \in \mathcal{F}$ *satisfying* $c[A \otimes B] = \{i\}$*, where* $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B\}$*.*

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