Slicely countably determined points in Banach spaces

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Joint work with Johann Langemets, Miguel Martín and Abraham Rueda Zoca
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Outline of the talk

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Notation

- X real or complex Banach space, X^* dual space
- S_X unit sphere, B_X closed unit ball
- $\operatorname{conv}(\cdot)$ convex hull, $\overline{\operatorname{conv}}(\cdot)$ closed convex hull

A slice of A (bounded convex $\subset X$) is a (nonempty) subset of the form

$$S(A, x^*, \alpha) := \{x \in A : \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \alpha\} \quad (x^* \in X^*, \alpha > 0)$$

SCD sets and spaces

Let $A \subset X$ be bounded and convex.

Definition (A. Avilés, V. Kadets, M. Martín, J. Merí, V. Shepelska (2010))

A sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A is determining for A, if one of the following equivalent conditions hold:

- if $B \subset A$ satisfies $B \cap V_n \neq \emptyset$ for every n, then $A \subset \overline{\text{conv}}(B)$;
- if $x_n \in V_n$ for every n, then $A \subset \overline{\text{conv}}(\{x_n \colon n \in \mathbb{N}\})$;
- if for every slice S of A, there is a V_m such that $V_m \subset S$.

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Definition (AKMMS (2010))

The set A is said to be slicely countably determined (an SCD set in short), if there exists a determining sequence of slices of A.

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Definition (AKMMS (2010))

The set A is said to be slicely countably determined (an SCD set in short), if there exists a determining sequence of slices of A.

Important! The definition of the SCD set implies the separability of the set.

Properties and examples of SCD sets

A point $a \in A$, where A is closed, bounded and convex, is called a denting point of A if for every $\varepsilon > 0$ there exists a slice S of A such that $a \in S$ and $\text{diam}(S) < \varepsilon$. We denote the set of denting points of A as $\text{dent}(A)$.

Example (AKMMS (2010))

If A is separable and dentable ($A = \overline{\text{conv}}(\text{dent}(A))$), then A is SCD.

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If X^* is separable, then every A is SCD.

Recall that X has the Daugavet property, if for every $x \in S_X$, slice S of B_X and $\varepsilon > 0$, there exists $y \in S$ such that $\|x - y\| > 2 - \varepsilon$.

Example (AKMMS (2010))

If X has the Daugavet property, then B_X is not SCD.

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Example (AKMMS (2010))

- If X is separable and has RNP (every closed, convex, bounded subset is dentable), then X is an SCD space.
- If X is a separable and X^* is RNP (X is Asplund), then X is an SCD space.
- If X is a separable Banach space which admits an equivalent renorming with the Daugavet property, then X is not SCD.

SCD spaces

Proposition

Let M be a separable metric space. Then $\mathcal{F}(M)$ has the RNP if and only if $\mathcal{F}(M)$ is an SCD space.

Proof.

($\Longrightarrow$). All separable spaces with the RNP are SCD spaces.

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($\Longleftarrow$). Assume that $\mathcal{F}(M)$ is not RNP. Then it must contain a subspace Y isomorphic to $L_1[0,1]$ (R. Aliaga, C. Gartland, C. Petitjean, A. Procházka (2022)).

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Problem

- Does every separable Banach space that is not SCD possess the Daugavet property in some equivalent norm?

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Definition (J. Langemets, L, M. Martín, A. Rueda Zoca (2024))

We say that a countable sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A is determining for point $a \in A$ if $a \in \overline{\text{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n.

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Proposition (LLMZ (2024))

For a sequence $\{V_n \colon n \in \mathbb{N}\}$ of subsets of A, the following conditions are equivalent:

(i) $\{V_n \colon n \in \mathbb{N}\}$ *is determining for a;*

(ii) *for every slice S of A with $a \in S$, there is $m \in \mathbb{N}$ such that $V_m \subset S$;*

(iii) *if $x_n \in V_n$ for every $n \in \mathbb{N}$, then $a \in \overline{\operatorname{conv}}(\{x_n \colon n \in \mathbb{N}\})$.*

SCD points

Definition (LLMZ (2024))

A point $a \in A$ is called a slicely countably determined point of A (an SCD point of A in short), if there exists a determining sequence of slices of A for the point a.

We denote the set of all SCD points of A by $\mathrm{SCD}(A)$.

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Proposition (LLMZ (2024))

The following statements hold:

1. *If A is an SCD set, then every $a \in A$ is an SCD point.*
2. *If every $a \in A$ is an SCD point and A is separable, then A is an SCD set.*
3. *$\mathrm{SCD}(A)$ is convex and closed. If A is balanced, then $\mathrm{SCD}(A)$ is balanced.*
4. *$\mathrm{SCD}(B_X) \neq \emptyset$ if and only if $0 \in \mathrm{SCD}(B_X)$.*

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Example (LLMZ (2024))

If X has RNP, then $\mathrm{SCD}(A) = A$ for any convex bounded subset A of X. However, we have $\mathrm{SCD}(B_{c_0(I)}) = \emptyset$, if I is uncountable.

Is it always either $\mathrm{SCD}(B_X) = B_X$ or $\mathrm{SCD}(B_X) = \emptyset$?

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A Banach space where $\mathrm{SCD}(B_X) = \{0\}$

Let (X_n) be Banach spaces. Consider $X := \left(\bigoplus_{n=1}^{\infty} X_n\right)_p$ endowed with the norm

$$\|x\| = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p}, \quad \text{where } x = (x_n)_{n=1}^{\infty} \text{ and } 1 < p < \infty.$$

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Theorem (LLMZ (2024))

If (X_n) is arbitrary, then $0 \in SCD(B_X)$.

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Proposition (LLMZ (2024))

Assume that $X := E \oplus_p Y$, where E has the Daugavet property, Y is arbitrary, and $1 < p < \infty$. If $(a, b) \in \mathrm{SCD}(B_X)$, then $a = 0$.

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Theorem (LLMZ (2024))

Consider the Banach space $X := \left(\bigoplus_{n=1}^{\infty} E_n\right)_p$, where $1 < p < \infty$ and E_n are spaces with the Daugavet property. Then $\mathrm{SCD}(B_X) = \{0\}$.

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SCD points in $\mathcal{F}(M)$

Theorem (LLMZ (2024))

Let M be a complete metric space and let $\mu \in S_{\mathcal{F}(M)}$. TFAE:

(i) μ *is an SCD point of* B_X.

(ii) $\mu \in \overline{\mathrm{conv}}(\mathrm{dent}(B_{\mathcal{F}(M)}))$.

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(ii) $\mu \in \overline{\text{conv}}(\text{dent}(B_{\mathcal{F}(M)}))$.

A point $x_0 \in B_X$ is a strongly exposed point if there is a $x^* \in X^*$ such that $x_0 \in \text{diam}(S(B_X, x^*, \alpha)) \to 0$ whenever $\alpha \to 0$.

Theorem (LLMZ (2024))

Let M be a compact metric space and let $\mu \in S_{\mathcal{F}(M)}$. TFAE:

(i) μ *is an SCD point of* B_X.

(ii) $\mu \in \overline{\text{conv}}(str.exp(B_{\mathcal{F}(M)}))$.

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