Fedorchuk compacta and LUR renormability

Todor Manev

Sofia University "St. Kliment Ohridski"

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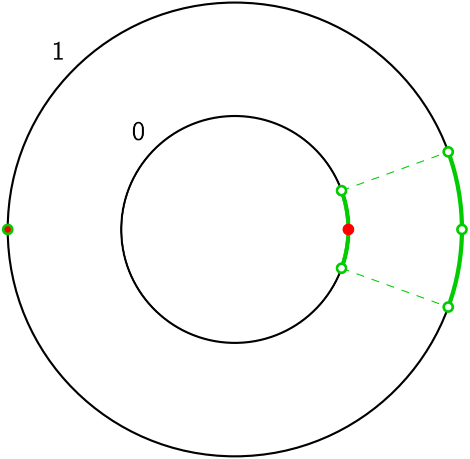
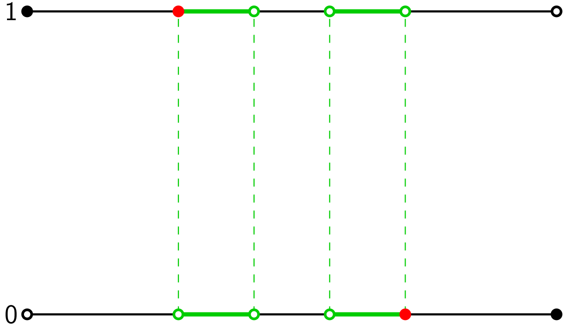
New Perspectives in Banach Spaces and Banach Lattices

CIEM Castro Urdiales

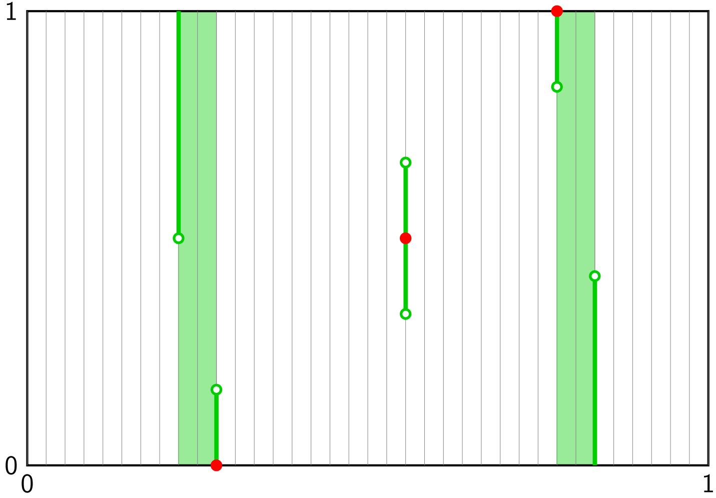
Double circle



Double arrow



Lexicographic square



Preliminaries

Definition

Let E be a Banach space. The norm on E is called *locally uniformly rotund (LUR)* if for any point x in the unit sphere S_E and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset S_E$ we have that

$$\lim_{n\to\infty} \left\| \frac{x + x_n}{2} \right\| = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} \|x - x_n\| = 0$$

Stability of LUR renormability of $C(K)$ spaces

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- Continuous images

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- Resolutions

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S. Watson, *The construction of topological spaces, planks and resolutions*, in: Recent progress in general topology, M. Hušek, J. van Mill (Eds.), vol. 20, North-Holland, 1992, 673–757. .

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- Y, X_y for $y \in Y$ - topological spaces.
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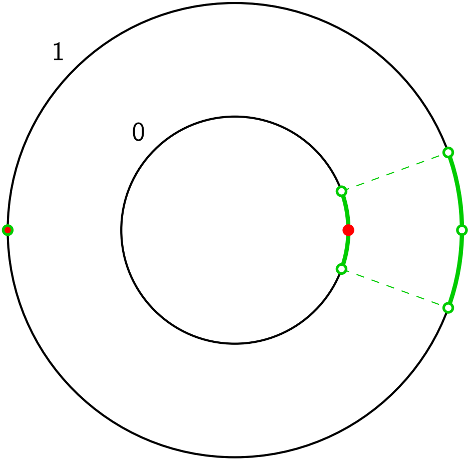
- $X := \left\{\{y\} \times X_y : y \in Y\right\}$.
- $\{y\} \times V \cup \left(\bigcup \left\{y' \times Y_{x'} : x' \in (U \cap h_y^{-1}(V))\right\}\right)$ for $U \subset Y, V \subset X_y$ open - basic open sets.

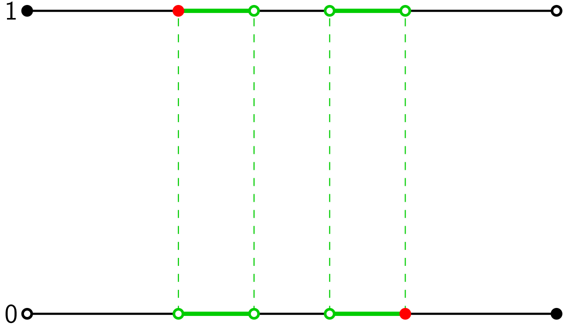
Double circle

$Y = S^1$, $X_y = \{0, 1\}$, $h_y(y') = 0 \ \forall y \neq y' \in Y$.



Double arrow

$Y = [0,1]$, $X_y = \{0,1\}\ \forall y \in Y$, $h_y(y') = 0$ if $y' < y$ and $h_y(y') = 1$ if $y' > y$.



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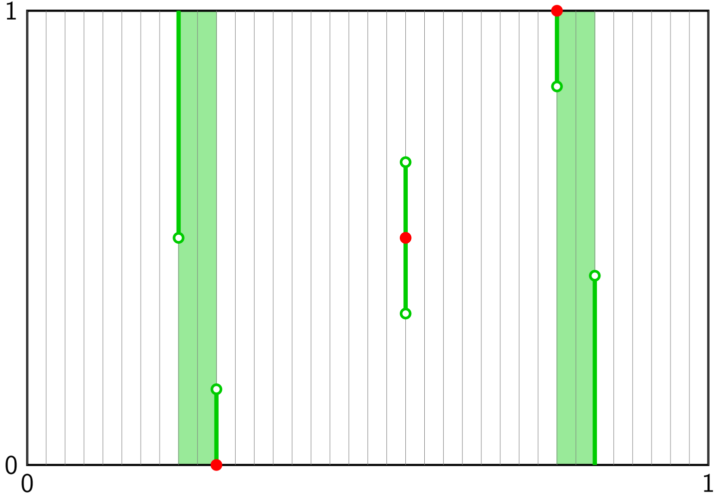


Definition (Fedorchuk ('68))

Let X and Y be topological spaces and $f : X \to Y$ a continuous mapping. f is called *fully closed* at $y \in Y$ if for any finite open cover $\{U_1, \dots, U_s\}$ of $f^{-1}(y)$ the set $\{y\} \cup \bigcup_{i=1}^{s} f^{\#}(U_i)$ is a neighborhood of y. A continuous surjective mapping is called *fully closed* if it is fully closed at every point of Y.

Preliminaries

Definition (Ivanov ('84))

Let $S = \{X_\alpha, \pi_\alpha^\beta, \alpha, \beta \in \mu\}$ be a continuous inverse system, where X_α are Hausdorff compacta, the neighboring bonding mappings $\pi_\alpha^{\alpha+1}$ are fully closed, X_0 is a point, and the fibers $\left(\pi_\alpha^{\alpha+1}\right)^{-1}(y)$ are metrizable compacta for all $\alpha + 1 \in \mu$ and all $y \in X_\alpha$. Then $\varprojlim S$ is called a *Fedorchuk compact* of *spectral height* μ, provided that the system is minimal.

Preliminaries

Theorem (Gul'ko, Ivanov, Shulikina, Troyanski ('20))

Let X be a Hausdorff compact admitting a fully closed projection π onto a metrizable compact Y. In addition, let the fibers $\pi^{-1}(y)$ be metrizable for all $y \in Y$. Then $C(X)$ admits an equivalent pointwise lower semicontinuous LUR norm.

Preliminaries

Theorem (M. (23'))

Let X and Y be Hausdorff compacta and π a fully closed mapping from X onto Y. Then $C(X)$ admits an equivalent τ_p-lower semicontinuous LUR norm provided that the spaces $C(Y)$ and $C\left(\pi^{-1}(y)\right)$ for $y \in Y$ admit equivalent τ_p-lsc LUR norms.

Corollary

$C(X)$ admits an equivalent pointwise-lower semicontinuous LUR norm whenever X is a Fedorchuk compact of finite spectral height.

Main result

Main Theorem

Let $X = \varprojlim\{X_n, \pi_n^k, n, k \in \omega\}$ *be a Fedorchuk compact of spectral height* ω *with the additional property that all bonding mappings* π_k^n *are fully closed. Then* $C(X)$ *admits an equivalent* τ_p*-lsc LUR norm.*

Preliminaries

Theorem (Moltó, Orihuella, Troyanski ('97))

Let E be a Banach space and F a norming subspace of its dual. Then E admits an equivalent $\sigma(E, F)$-lower semicontinuous LUR norm if and only if for any $\epsilon > 0$ there exists a countable decomposition $X = \bigcup_{n \in \mathbb{N}} X_n$ such that for all $n \in \mathbb{N}$ and $x \in X_n$ there exists H an open half space containing x and satisfying:

$$\|\cdot\|\text{-}\operatorname{diam}(X_n \cap H) < \epsilon.$$

Notation

Let X and Y be topological spaces and π a continuous mapping from X onto Y. If $M \subset Y$, by Y^M we will denote the quotient space corresponding to the following equivalence classes:

$$[x] \quad = \quad \begin{cases} x, & x \in \pi^{-1}(M); \\ \pi^{-1}\left(\pi(x)\right), & \pi(x) \in Y \setminus M. \end{cases}$$

We will denote the corresponding quotient mapping from X to Y^M by p^M and by $\pi^M : Y^M \to Y$ the unique mapping such that $\pi = \pi^M \circ p^M$.

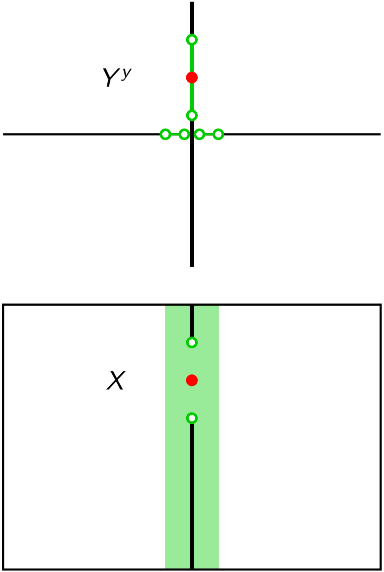
Preliminaries

Proposition (Fedorchuk ('06))

Let X and Y be compact topological spaces and $f : X \to Y$ a closed map. Then the following are equivalent:

1. *f is fully closed.*
2. *If $F_1, F_2 \subset X$ are closed and disjoint, the set $f(F_1) \cap f(F_2)$ is finite.*
3. *If $U \subset X$ is open and $y \in Y$, the set $U^y := \left(f^{-1}(y) \cap U\right) \cup f^{-1}\left(f^{\#}(U)\right)$ is open.*
4. *For any $M \subset Y$, the space Y^M is Hausdorff.*

$\mathbb{R}^2$ square



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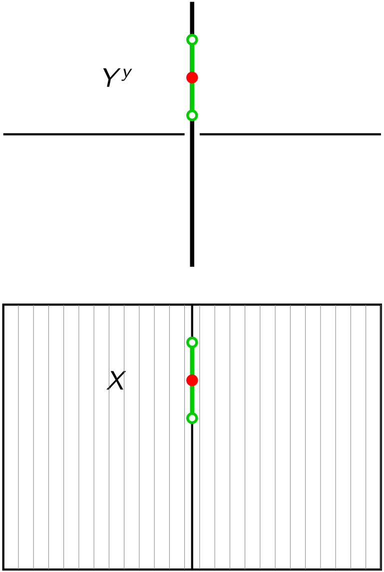

Preliminaries

Proposition (Gulko, Ivanov, Shulikina, Troyanski ('20))

Let $\pi : X \to Y$ be a continuous surjective mapping between Hausdorff compacta. Then π is fully closed if and only if for all $f \in C(X)$ we have $\left(\operatorname{osc}_{\pi^{-1}(y)} f : y \in Y\right) \in c_0(Y)$.

Idea of the proof

$M_n^{f,\epsilon} := \{x \in X_n : \operatorname{osc}_{\pi_n^{-1}(x)} f > \epsilon\}.$

$$C(X) = \bigcup_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}^{m-1}} \bigcup_{r\in\mathbb{N}} \bigcup_{q\in\mathbb{Q}^{\sum k_i}} \bigcup_{p\in\mathbb{N}} E_{m,k,r,q,p} :$$

$$m = \min\left\{n \in \mathbb{N} : M_n^{f,\epsilon} = \emptyset\right\}; \tag{1}$$

$$\left|M_i^{f,\epsilon}\right| = k_i; \quad i = 1, \dots, m-1; \tag{2}$$

$$\min\{\operatorname{osc}_{\pi_n^{-1}(x)} f :\ n \in \mathbb{N}, x \in M_n^{f,\epsilon}\} - \epsilon > \frac{1}{r}; \tag{3}$$

$$\operatorname{osc}_{\pi_n^{-1}(x_j)} f \in \left(q_j - \frac{1}{3r\sum_{i=1}^{m-1} k_i}, q_j + \frac{1}{3r\sum_{i=1}^{m-1} k_i}\right); \tag{4}$$

$$f \in B_\epsilon(h_p), \quad \overline{\{h_p\}_{p\in\mathbb{N}}}^{\|\cdot\|} = C(K_f^\epsilon). \tag{5}$$

Idea of the proof

Let $M_n \subset X_n$ be subsets satisfying $\pi_k^n(M_n) \subset M_k$.

$$K_0 := X_0, \quad K_1 := X_1, \quad K_2 := (X_1)_{\pi_1^2}^{M_1}$$
$$K_{n+1} := (K_n)_{q_n \circ \pi_n^{n+1}}^{q_n(M_n)};$$
$$\lambda_n^{n+1} := \left(q_n \circ \pi_n^{n+1}\right)^{q_n(M_n)}.$$

q_{n+1} is the unique map giving $q_n \circ \pi_n^{n+1} = \lambda_n^{n+1} \circ q_{n+1}$.
$K := \varprojlim\{K_n, \lambda_k^n, n, k \in \mathbb{N}\}$. $q(x) := \{q_n(x_n)\}_{n \in \mathbb{N}}$.

Idea of the proof

Proposition ([Fed06])

Let $\pi : X \to Y$ be a fully closed mapping between Hausdorff compacta. Then X is metrizable if and only if the following conditions hold:

- *Y is metrizable;*
- *All the fibers $\pi^{-1}(y)$ are metrizable;*
- *The set of nontrivial fibers $\left\{y \in Y : \left|\pi^{-1}(y)\right| \geq 2\right\}$ is countable.*

Questions

Question

Let X be a Fedorchuk compact of countable spectral height. Does $C(X)$ admit an equivalent pointwise-lower semicontinuous LUR norm?

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Let $S = \{X_\beta, \pi^\alpha_\beta, \gamma\}$ be a continuous inverse system, where γ is a countable ordinal, X_0 is a singleton, the neighboring bonding mappings $\pi^{\beta+1}_\beta$ are fully closed, and for any $\beta + 1 < \gamma$ and any $y \in X_\beta$ the space $C\left((\pi^{\beta+1}_\beta)^{-1}(y)\right)$ is LUR renormable. Let X be the limit $\varprojlim S$. Does $C(X)$ admit an equivalent LUR norm?

Thank you for the attention!