On some measures related to ideals of multilinear operators and interpolation

Antonio Manzano
(Universidad de Burgos, Spain)

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and E. A. Sánchez-Pérez (Universidad Politécnica de Valencia, Spain).

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• Astala introduced in 1980 the so-called outer measure $\gamma_{\mathcal{I}}$ of $T \in \mathcal{L}(E;F)$, associated to an arbitrary operator ideal $\mathcal{I}$ as

$$\gamma_{\mathcal{I}}(T) = \gamma_{\mathcal{I}}(T : E \to F) := \inf \{\varepsilon > 0 \,:\, T(B_E) \subseteq \varepsilon B_F + R(B_Z), \text{ for some Banach space } Z \text{ and operator } R \in \mathcal{I}(Z;F)\}.$$

When $\mathcal{I}$ is a closed surjective ideal, it holds that

$$T \in \mathcal{I}(E;F) \quad \text{if and only if} \quad \gamma_{\mathcal{I}}(T) = 0.$$

K. Astala, *On measures of non-compactness and ideal variations in Banach spaces.* Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes **29** (1980), 1–42.

• Tylli defined in 1995 the inner measure $\beta_{\mathcal{I}}$ of $T \in \mathcal{L}(E;F)$, related to a general operator ideal $\mathcal{I}$, by

$$\beta_{\mathcal{I}}(T) = \beta_{\mathcal{I}}(T : E \to F) := \inf \{\varepsilon > 0 \,:\, \text{there are a Banach space } Z \text{ and an operator } R \in \mathcal{I}(E;Z) \text{ s.t. } \|Tx\|_F \leq \varepsilon \|x\|_E + \|Rx\|_Z,\ x \in E\}.$$

If $\mathcal{I}$ is a closed injective ideal, then

$$T \in \mathcal{I}(E;F) \quad \text{if and only if} \quad \beta_{\mathcal{I}}(T) = 0.$$

H. O. Tylli, *The essential norm of an operator is not self–dual.* Israel J. Math. **91** (1995), 93–110.

Significant **examples of closed surjective injective operator ideals** are: **compact operators**, **weakly compact operators**, **Banach-Saks operators**, **Rosenthal operators** and **Asplund operators**.

- For $\mathcal{I} = \mathcal{K}$, the ideal of compact operators,

$$\gamma_{\mathcal{K}}(T) = \gamma(T) := \inf\Big\{\varepsilon > 0 : T(B_E) \subseteq \bigcup_{k=1}^{m} \{y_k + \varepsilon B_F\}, y_k \in F, m \in \mathbb{N}\Big\}.$$

and

$$\beta_{\mathcal{K}}(T) = \|T\|_m := \inf\Big\{\varepsilon > 0 : \text{there is } M \text{ of } E \text{ with } \operatorname{codim}(M) < \infty \text{ such that } \|Tx\|_F \leq \varepsilon\|x\|_E, \text{ for any } x \in M\Big\}.$$

- When $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators,

$$\gamma_{\mathcal{W}}(T) = w(T) := \inf\{\varepsilon > 0 : T(B_E) \subseteq \varepsilon B_F + W, \, W \subseteq F \text{ weakly compact}\}.$$

The research of different properties of $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$, specially their behaviour under interpolation, has been very useful and it has enabled to give an unified point of view for previous results on some operator ideals that are closed and surjective or closed and injective.

THEOREM (Cobos, M. and Martínez, 1999)

Let $\mathcal{I}$ be an operator ideal. Assume that $\bar{E} = (E_0, E_1)$ is a Banach couple and F is a Banach space. Let E be an intermediate space of class $C_K(\theta, \bar{E})$ with constant C. If $T : E_0 + E_1 \to F$ is a continuous linear operator,

$$\gamma_{\mathcal{I}}(T : E \to F) \leq C(1-\theta)^{\theta-1}\theta^{-\theta}\gamma_{\mathcal{I}}(T : E_0 \to F)^{1-\theta}\gamma_{\mathcal{I}}(T : E_1 \to F)^{\theta}.$$

F. Cobos, A. M. and A. Martínez, *Interpolation theory and measures related to operator ideals.* Quarterly J. Math. **50** (1999), 401–416.

This theorem allows to deduce previous results regarding interpolation of *compact* operators, *weakly compact* operators, *strictly co-singular* operators or *Rosenthal* operators. In fact, **it provides a quantitative version of Heinrich's result:**

THEOREM (Heinrich, 1980)

Let $\mathcal{I}$ be a closed surjective operator ideal. Let $\bar{E} = (E_0, E_1)$ be a Banach couple and let F be a Banach space. Suppose that E is of class $C_K(\theta, \bar{E})$. If $T : E_0 + E_1 \to F$ is a continuous linear operator and

$$T : E_r \to F \text{ belongs to the ideal } \mathcal{I} \text{ for } r = 0 \text{ or } r = 1,$$

then

$$T : E \to F \text{ belongs to } \mathcal{I}.$$

S. Heinrich, *Closed operator ideals and interpolation.* J. Funct. Anal. **35** (1980),

Theorem (Cobos, M. and Martínez, 1999)

Let $\mathcal{I}$ be an operator ideal. Assume that E is a Banach space and $\bar{F} = (F_0, F_1)$ is a Banach couple. Let F be an intermediate space of class $C_J(\theta, \bar{F})$ with constant C. If $T : E \to F_0 \cap F_1$ is a continuous linear operator,

$$\beta_{\mathcal{I}}(T : E \to F) \leq C\beta_{\mathcal{I}}(T : E \to F_0)^{1-\theta}\beta_{\mathcal{I}}(T : E \to F_1)^{\theta}.$$

F. Cobos, A. M. and A. Martínez, *Interpolation theory and measures related to operator ideals.* Quarterly J. Math. **50** (1999), 401–416.

This theorem recovers known results on interpolation of *compact* operators, *weakly compact* operators, *strictly singular* operators or *Rosenthal* operators. It provides a quantitative version of Heinrich's result:

Theorem (Heinrich, 1980)

Let $\mathcal{I}$ be a closed injective operator ideal. Let E be a Banach space and $\bar{F} = (F_0, F_1)$ be a Banach couple. Suppose that F is an intermediate space of class $C_J(\theta, \bar{F})$. If $T : E \to F_0 \cap F_1$ is a continuous linear operator and

$$T : E \to F_r \text{ belongs to the ideal } \mathcal{I} \text{ for } r = 0 \text{ or } r = 1,$$

then

$$T : E \to F \text{ belongs to } \mathcal{I}.$$

S. Heinrich, *Closed operator ideals and interpolation.* J. Funct. Anal. **35** (1980),

Since the paper by

A. Pietsch, *Ideals of multilinear functionals (designs of a theory).* In: Proc. Second Int. Conf. on Operator Algebras, Ideals and Their Applications in Theoretical Physics, Teubner-Texte Math. **67** (1983), 185–199.

was published, the study of ideals of multilinear operators between Banach spaces has been inspired by certain fruitful ideas and techniques used in linear operator theory.

Two examples that show an interplay between the linear and multilinear cases are the n-ideals (or ideals of n-linear operators)

$$[\mathcal{I}_1, \ldots, \mathcal{I}_n] \quad \text{and} \quad \mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n),$$

generated by (linear) operator ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$.

A natural question is if it is possible to consider similar notions (in some sense) to $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$ in the setting of multilinear operators. We are interested in this problem. **We shall introduce two multilinear measures that can be considered as a generalization of $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$.** These new measures shall characterize the operators that belong to (depending on the case) a closed surjective ideal, or a closed injective ideal, of multilinear operators as those having measure equal to zero. **We shall also obtain some interpolation formulas for these new measures and so interpolation results involving ideals of multilinear operators**.

• Given $E_1, \ldots, E_n$ Banach spaces, $E_1 \times \cdots \times E_n$ denotes their product equipped with the standard norm $\|(x_1, \ldots, x_n)\| = \max\{\|x_i\|_{E_i} : i = 1, \ldots, n\}$. If F is another Banach space, $\mathcal{L}(E_1, \ldots, E_n; F)$ stands for the *space of all continuous n-linear operators* $T : E_1 \times \cdots \times E_n \to F$, with the norm

$$\|T\|_{\mathcal{L}(E_1,\ldots,E_n;F)} := \sup\{\|T(x_1, \ldots, x_n)\|_F \ : \ x_1 \in B_{E_1}, \ldots, x_n \in B_{E_n}\},$$

where B_{E_i} is the closed unit ball of E_i $(i = 1, \ldots, n)$. In particular, $\mathcal{L}(E; F)$ is the Banach space of all continuous linear operators from E into F.

• Let $n \in \mathbb{N}$ be fixed. An *n-ideal*, or an *ideal of n-linear operators*, is a class $\mathcal{M}_n$ of n-linear maps such that for any Banach spaces $E_1, \ldots, E_n, F$, the components $\mathcal{M}_n(E_1, \ldots, E_n; F) := \mathcal{L}(E_1, \ldots, E_n; F) \cap \mathcal{M}_n$ satisfy:

(i) $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ that contains the n-linear maps of finite type.

(ii) If $R \in \mathcal{L}(F; H)$, $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$ and $S_j \in \mathcal{L}(G_j; E_j)$, for $j = 1, \ldots, n$, then $R \circ T \circ (S_1, \ldots, S_n) \in \mathcal{M}_n(G_1, \ldots, G_n; H)$.

If for each $n \in \mathbb{N}$, $\mathcal{M}_n$ is an *n-ideal*, the class $\mathcal{M} := \cup_{n=1}^{\infty} \mathcal{M}_n$ is called *multi-ideal* or *ideal of multilinear operators*.

For $n = 1$, the previous definition yields the notion of *(linear) operator ideal* in the sense of Pietsch.

A. PIETSCH, *"Operator Ideals"*. NORTH–HOLLAND, AMSTERDAM, 1980.

We use the letter $\mathcal{I}$ to denote a *(linear) operator ideal* instead of $\mathcal{M}_1$ or $\mathcal{I}_1$.

- Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be operator ideals:

i) The n-ideal $[\mathcal{I}_1, \ldots, \mathcal{I}_n]$ is defined as follows (the symbol $\overset{[j]}{\cdots}$ means that the j-th term does not appear): let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$T \in [\mathcal{I}_1, \ldots, \mathcal{I}_n](E_1, \ldots, E_n; F) \text{ if and only if } T_j \in \mathcal{I}_j(E_j; \mathcal{L}(E_1, \overset{[j]}{\ldots}, E_n; F)),$$

for any $j = 1, 2, \ldots, n$, where $T_j : E_j \to \mathcal{L}(E_1, \overset{[j]}{\ldots}, E_n; F)$ is given by

$$T_j(x_j)(x_1, \overset{[j]}{\ldots}, x_n) := T(x_1, \ldots, x_n),\ x_1 \in E_1, \ldots, x_n \in E_n\ (j = 1, \ldots, n).$$

ii) The n-ideal $\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ is defined as follows: Let $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$T \in \mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n)(E_1, \ldots, E_n; F) \text{ if and only if } T = S \circ (R_1, \ldots, R_n),$$

for some $R_j \in \mathcal{I}_j(E_j; G_j)$ $(j = 1, 2, \ldots, n)$ and $S \in \mathcal{L}(G_1, \ldots, G_n; F)$.

• It holds that $\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n) \subseteq [\mathcal{I}_1, \ldots, \mathcal{I}_n]$ and, in general, these n-ideals do not coincide. However, if $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are closed surjective injective ideals, then

$$\mathcal{L}(\mathcal{I}_1, \ldots, \mathcal{I}_n) = [\mathcal{I}_1, \ldots, \mathcal{I}_n].$$

M. González and J. M. Gutiérrez, *Injective factorization of holomorphic mappings.* Proc. Amer. Math. Soc. **127** (1999), 1715–1721; and *Erratum to "Injective factorization of holomorphic mappings".* Proc. Amer. Math. Soc. **129** (2001), 1255–1256.

H. A. Braunss and H. Junek, *Factorization of injective ideals by interpolation.* J. Math. Anal. Appl. **297** (2004), 740–750.

- We say that an n-ideal $\mathcal{M}_n$ is:

a) *closed* when each component $\mathcal{M}_n(E_1, \dots, E_n; F)$ is a closed subspace in $\mathcal{L}(E_1, \dots, E_n; F)$.

b) *surjective* whenever, for all $E_1, \dots, E_n, F$ and every $T \in \mathcal{L}(E_1, \dots, E_n; F)$, $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ if there are $Z_1, \dots, Z_n$ and $P \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$ such that $T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq P(B_{Z_1} \times \cdots \times B_{Z_n})$.

c) *injective* if for any $E_1, \dots, E_n, F$, each $T \in \mathcal{L}(E_1, \dots, E_n; F)$ and every isometric embedding $J \in \mathcal{L}(F; G)$, it holds that $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ whenever $J \circ T \in \mathcal{M}_n(E_1, \dots, E_n; G)$.

For $n = 1$ are recovered the corresponding notions for (linear) operator ideals.

- When every ideal $\mathcal{I}_j$ is closed and surjective (resp. closed and injective), $\mathcal{M}_n = [\mathcal{I}_1, \dots, \mathcal{I}_n]$ is also closed and surjective (resp. closed and injective).

G. Botelho, P. Galindo and L. Pellegrini, *Uniform approximation on ideals of multilinear mappings.* Math. Scand. **106** (2010), 301–319.

A. M., P. Rueda and E. A. Sánchez-Pérez, *Closed injective ideals of multilinear operators, related measures and interpolation.* Math. Nachr. **293** (2020), 510–532.

A. M., P. Rueda and E. A. Sánchez-Pérez, *Closed surjective ideals of multilinear operators and interpolation.* Banach J. Math. Anal. **15** (2021), article No. 27.

Theorem 1

Let $\mathcal{M}_n$ be an n-ideal which is closed and surjective. If $T \in \mathcal{L}(E_1, \dots, E_n; F)$, the following assertions are equivalent:

(a) $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$.

(b) For each $\varepsilon > 0$, there are Banach spaces $Z_1, \dots, Z_n$ and an n-linear operator $R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)$ such that

$$T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq \varepsilon B_F + R(B_{Z_1} \times \cdots \times B_{Z_n}).$$

Definition

Let $\mathcal{M}_n$ be an n-ideal. For $T \in \mathcal{L}(E_1, \dots, E_n; F)$, let

$$\gamma_{\mathcal{M}_n}(T) = \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) :=$$
$$\inf\Big\{\varepsilon > 0 \,:\, T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq \varepsilon B_F + R(B_{Z_1} \times \cdots \times B_{Z_n}), \text{ for some Banach spaces } Z_j \text{ and an } n\text{-linear operator } R \in \mathcal{M}_n(Z_1, \dots, Z_n; F)\Big\}.$$

A. M., P. Rueda and E. A. Sánchez-Pérez, *Closed surjective ideals of multilinear operators and interpolation*. Banach J. Math. Anal. **15** (2021), article No. 27.

Definition

Let $\mathcal{M}_n$ be an n-ideal. For $T \in \mathcal{L}(E_1, \ldots, E_n; F)$, let

$$\gamma_{\mathcal{M}_n}(T) = \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) := \inf\Big\{\varepsilon > 0 \,:\, T(B_{E_1} \times \cdots \times B_{E_n}) \subseteq \varepsilon B_F + R(B_{Z_1} \times \cdots \times B_{Z_n}), \text{ for some Banach spaces } Z_j \text{ and an } n\text{-linear operator } R \in \mathcal{M}_n(Z_1, \ldots, Z_n; F)\Big\}.$$

- For a closed surjective n-ideal $\mathcal{M}_n$,

$$T \in \mathcal{M}_n \text{ if and only if } \gamma_{\mathcal{M}_n}(T) = 0.$$

- It is obvious that $\gamma_{\mathcal{M}_n}$ is precisely Astala's measure if $n = 1$.

Theorem (Botelho, Galindo and Pellegrini, 2010)

Let $\mathcal{M}_n$ be an n-ideal which is closed and injective. Let $T \in \mathcal{L}(E_1, \dots, E_n; F)$. The following assertions are equivalent:

(a) $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$.

(b) For every $\varepsilon > 0$, there are a Banach space Z and an n-linear operator $R \in \mathcal{M}_n(E_1, \dots, E_n; Z)$ such that

$$\Big\| \sum_{j=1}^{m} T(x_1^j, \dots, x_n^j) \Big\|_F \leq \varepsilon \sum_{j=1}^{m} \|x_1^j\|_{E_1} \cdots \|x_n^j\|_{E_n} + \Big\| \sum_{j=1}^{m} R(x_1^j, \dots, x_n^j) \Big\|_Z,$$

for all $m \in \mathbb{N}$ and any $x_1^j \in E_1, \dots, x_n^j \in E_n, j = 1, \dots, m$.

G. Botelho, P. Galindo and L. Pellegrini, *Uniform approximation on ideals of multilinear mappings.* Math. Scand. **106** (2010), 301–319.

- Inspired by this theorem due to Botelho, Galindo and Pellegrini, we define:

Definition

Let $\mathcal{M}_n$ be an n-ideal. For $T \in \mathcal{L}(E_1, \ldots, E_n; F)$,

$$\beta_{\mathcal{M}_n}(T) = \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) :=$$

$$\inf \Big\{ \varepsilon > 0 : \text{there are a space } Z \text{ and an operator } R \in \mathcal{M}_n(E_1, \ldots, E_n; Z) \text{ s.t.}$$

$$\Big\| \sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j) \Big\|_F \leq \varepsilon \sum_{j=1}^{m} \|x_1^j\|_{E_1} \cdots \|x_n^j\|_{E_n} + \Big\| \sum_{j=1}^{m} R(x_1^j, \ldots, x_n^j) \Big\|_Z,$$

$$\text{for all } m \in \mathbb{N} \text{ and any } x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m \Big\}.$$

A. M., P. Rueda and E. A. Sánchez-Pérez, *Closed injective ideals of multilinear operators, related measures and interpolation.* Math. Nachr. **293** (2020), 510–532.

- For a closed injective n-ideal $\mathcal{M}_n$,

$$T \in \mathcal{M}_n \text{ if and only if } \beta_{\mathcal{M}_n}(T) = 0.$$

- It holds that $\beta_{\mathcal{M}_n}$ coincides with Tylli's measure when $n = 1$.

• Let $\bar{A} = (A_0, A_1)$ be a Banach couple; i.e., A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. Then, the sum and intersection spaces

$$\Sigma(\bar{A}) := A_0 + A_1 \quad \text{and} \quad \Delta(\bar{A}) := A_0 \cap A_1,$$

are Banach spaces when endowed with the norms $K(1, \cdot)$ and $J(1, \cdot)$, respectively. For $t > 0$, the K-functional and the J-functional are given by

$$K(t, a; A_0, A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, \ a_i \in A_i\}, \ a \in \Sigma(\bar{A}).$$

$$J(t, a; A_0, A_1) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \ a \in \Delta(\bar{A}).$$

A Banach space A is called intermediate space with respect to $\bar{A} = (A_0, A_1)$ if the following continuous inclusions hold:

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1.$$

An intermediate space A with respect to $\bar{A}$ is said to be *of class* $C_K(\theta; \bar{A})$ (resp. *of class* $C_J(\theta; \bar{A})$), where $0 < \theta < 1$, if there is $C > 0$ such that for all $t > 0$ and any $a \in A$ (resp. $a \in A_0 \cap A_1$),

$$K(t, a; A_0, A_1) \leq Ct^{\theta}\|a\|_A \quad (\text{resp. } \|a\|_A \leq Ct^{-\theta} J(t, a, A_0, A_1)).$$

The real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ and the complex interpolation space $\bar{A}_{[\theta]} = (A_0, A_1)_{[\theta]}$ are examples of spaces of class $C_K(\theta; \bar{A})$ and $C_J(\theta; \bar{A})$.

- Let $0 < \theta < 1$, $1 \leq q \leq \infty$, the real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ is formed by all elements $a \in A_0 + A_1$ with a finite norm

$$\|a\|_{\theta,q} := \left(\int_0^\infty \left(t^{-\theta} K(t, a; A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q}, \quad \text{when } 1 \leq q < \infty,$$

$$\|a\|_{\theta,q} := \sup_{t>0} t^{-\theta} K(t, a; A_0, A_1), \quad \text{when } q = \infty.$$

The space $(A_0, A_1)_{\theta,q}$ can be also described using the J-functional. Namely, an element $a \in (A_0, A_1)_{\theta,q}$ if and only if

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad \text{(convergence in } \Sigma(\bar{A})),$$

where $u(t)$ is a measurable function with values in $\Delta(\bar{A})$ and

$$\left(\int_0^\infty \left(t^{-\theta} J(t, u(t); A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} < \infty, \quad \text{if } 1 \leq q < \infty, \qquad (1)$$

(integral is replaced by supremum when $q = \infty$)
The infimum of (1) taken over all possible representations $u(t)$ gives an equivalent norm to $\| \cdot \|_{\theta, q}$.

• Given a Banach couple $\bar{A} = (A_0, A_1)$, let $\mathcal{F}(\bar{A})$ be the set of functions f defined on $S = \{z \in \mathbb{C} : 0 \leq \text{ Re } z \leq 1\}$ with values in $A_0 + A_1$, such that:

(a) f is bounded and continuous on S, and analytic on the open strip $S_0 = \{z \in \mathbb{C} : 0 < \text{ Re } z < 1\}$,

(b) the functions $t \longrightarrow f(j + it)$ $(j = 0, 1)$ are continuous from $\mathbb{R}$ into A_j, and tend to zero as $|t| \to \infty$.

$\mathcal{F}(\bar{A})$ is a Banach space equipped with

$$\|f\|_{\mathcal{F}} = \max\Big\{ \sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1} \Big\}.$$

The complex interpolation space $\bar{A}_{[\theta]} = (A_0, A_1)_{[\theta]}$, $0 < \theta < 1$, consists in

$$a \in A_0 + A_1 \text{ such that } a = f(\theta), \text{ for some } f \in \mathcal{F}(\bar{A}).$$

$(A_0, A_1)_{[\theta]}$ is a Banach space with the norm

$$\|a\|_{[\theta]} = \inf\Big\{ \|f\|_{\mathcal{F}} : f(\theta) = a,\ f \in \mathcal{F}(\bar{A}) \Big\}.$$

Theorem 2

Let $\mathcal{M}_n$ be an n-ideal. Let $E_1, \dots, E_n, F$ be Banach spaces. Take any $i = 1, \dots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple. Assume X is of class $\mathcal{C}_K(\theta, \bar{X})$ with constant C. If $T \in \mathcal{L}(E_1, \dots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \dots, E_n; F)$,

$$\begin{aligned}&\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F)\\ &\leq C(1-\theta)^{\theta-1}\theta^{-\theta}\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_0 \times E_{i+1} \times \cdots \times E_n \to F)^{1-\theta} \cdot \\ &\quad \cdot \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times X_1 \times E_{i+1} \times \cdots \times E_n \to F)^{\theta}.\end{aligned}$$

Corollary 3

Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $E_1, \dots, E_n, F$ be Banach spaces. Take any $i = 1, \dots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple. Assume X is of class $\mathcal{C}_K(\theta, \bar{X})$. If $T \in \mathcal{L}(E_1, \dots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \dots, E_n; F)$, then

$$T \in \mathcal{M}_n(E_1, \dots, E_{i-1}, X, E_{i+1}, \dots, E_n; F)$$

whenever, for $r = 0$ or $r = 1$, $T \in \mathcal{M}_n(E_1, \dots, E_{i-1}, X_r, E_{i+1}, \dots, E_n; F)$.

In particular, **for $n = 1$, Theorem 2 allows to deduce the result of Cobos, M. and Martínez** established in the linear case. **Corollary 3 constitutes a multilinear version of Heinrich's result on closed surjective operator ideals.**

Theorem 4

Let $\mathcal{M}_n$ be an n-ideal, let $E_1, \ldots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$,

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq C\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F_0)^{1-\theta} \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F_1)^{\theta}.$$

Corollary 5

Let $\mathcal{M}_n$ be a closed injective n-ideal. Let $\bar{F} = (F_0, F_1)$ be a Banach couple and let F be of class $\mathcal{C}_J(\theta, \bar{F})$. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$, it holds that

$$T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$$

whenever $T \in \mathcal{M}_n(E_1, \ldots, E_n; F_r)$, for $r = 0$ or $r = 1$.

Theorem 4 recovers when $n = 1$ the aforementioned **result of Cobos, M. and Martínez**. **Corollary 5 can be read as a version in the multilinear case of Heinrich's result on closed injective operator ideals.**

According to Counterexamples 2.5 and 2.4 shown by Beucher in

O. J. Beucher, *On interpolation of strictly (co)-singular linear operators.* Proc. Royal Soc. Edinburgh Section A Mathematics **112A** (1989), 263–269.

we note that **even for $n = 1$ an estimate as that given in Theorem 2 does not hold in general** if $T \in \mathcal{L}(E; \Delta(\bar{F}))$. Likewise, **even for $n = 1$ a similar result to Theorem 4 does not hold in general** for $T \in \mathcal{L}(\Sigma(\bar{E}); F)$. Let us see the second assertion:

• Take $\mathcal{I} = \mathcal{S}$, the ideal of *strictly singular operators*, which is closed and injective. Let $\bar{E} = (L_\infty[0,1], L_1[0,1])$, $F = L_1[0,1]$ and let T be the identity operator. Due to a result proved by Grothendieck,

$$T : L_\infty[0,1] \to L_1[0,1] \text{ belongs to } \mathcal{I}.$$

But as it pointed out by Beucher, if $0 < \theta = 1/p < 1$, the operator

$$T : (L_\infty[0,1], L_1[0,1])_{\theta,p} = L_p[0,1] \to L_1[0,1]$$

does not belong to the ideal $\mathcal{I}$ since, according to Khintchine's inequality, the span of the Rademacher functions in $L_p[0,1]$ and $L_1[0,1]$ is isomorphic to ℓ_2. Thus, the restriction of the identity operator T to this subspace of $L_p[0,1]$ is an isomorphism into $L_1[0,1]$.

The next theorem provides an estimate for the measure $\gamma_{\mathcal{M}_n}$ of the interpolated operator in terms of the measures of the restrictions

$$T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F$$

and

$$T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F.$$

This is interesting since sometimes the known information about the operator refers to such extreme restrictions.

Theorem 6

Let $\mathcal{M}_n$ be an n-ideal. Let $E_1, \ldots, E_n, F$ be Banach spaces. Take any $i = 1, \ldots, n$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple. Assume X of class $\mathcal{C}_K(\theta, \bar{X})$ with constant C. If $T \in \mathcal{L}(E_1, \ldots, E_{i-1}, \Sigma(\bar{X}), E_{i+1}, \ldots, E_n; F)$,

$$\begin{aligned}\gamma_{\mathcal{M}_n}&(T : E_1 \times \cdots \times E_{i-1} \times X \times E_{i+1} \times \cdots \times E_n \to F) \\ &\leq 4C\gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Delta(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)^{\Theta} \cdot \\ &\qquad \cdot \gamma_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_{i-1} \times \Sigma(\bar{X}) \times E_{i+1} \times \cdots \times E_n \to F)^{1-\Theta},\end{aligned}$$

where $\Theta = \min\{\theta, 1-\theta\}$.

Using Theorem 6 and iteration, we get the next result for general closed surjective n-ideals and operators from $\Sigma(\bar{E}_1) \times \cdots \times \Sigma(\bar{E}_n)$ into F, with $\bar{E}_1, \ldots, \bar{E}_n$ arbitrary Banach couples.

Corollary 7

Let $\mathcal{M}_n$ be a closed surjective n-ideal. Let $\bar{E}_j = (E_{0j}, E_{1j})$ be a Banach couple, $j = 1, \ldots n$, and let F be a Banach space. Suppose that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j), j = 1, \cdots, n$. For any $T \in \mathcal{L}(\Sigma(\bar{E}_1), \ldots, \Sigma(\bar{E}_n); F)$,

$$T \in \mathcal{M}_n(E_1, \ldots, E_n; F) \text{ if and only if } T \in \mathcal{M}_n(\Delta(\bar{E}_1), \ldots, \Delta(\bar{E}_n); F).$$

Corollary 7 can be applied to $\mathcal{M}_n = [\mathcal{I}_1, \ldots, \mathcal{I}_n]$, where $\mathcal{I}_1, \ldots, \mathcal{I}_n$ are closed surjective operator ideals. In particular, **when $\mathcal{I}_1 = \cdots = \mathcal{I}_n = \mathcal{I}$ and $\mathcal{I}$ is any of the following closed surjective ideals**: **compact operators**, **weakly compact operators**, **strictly co-singular operators**, **Rosenthal operators**, **Banach-Saks operators**, or **Asplund operators**, **we obtain an extension to the multilinear case of previous interpolation results** for these operator ideals.

We also establish (in the dual situation) an estimate for the measure $\beta_{\mathcal{M}_n}$ of the interpolated operator in terms of the measures of extreme restrictions

$$T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}) \quad \text{and} \quad T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}).$$

Theorem 8

Let $\mathcal{M}_n$ be an n-ideal, let $E_1, \ldots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For $T \in \mathcal{L}(E_1, \ldots, E_n; \Delta(\bar{F}))$,

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq 4C\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta}\, \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta},$$

where $\Theta = \min\{\theta, 1-\theta\}$.

Proof

Let $\eta > 0$. Take any $t \geq 1$ such that

$$t^{-\theta} \leq \eta \quad \text{and} \quad t^{\theta-1} \leq \eta. \tag{2}$$

Let $\sigma > \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))$. It is possible to find a Banach space H and an operator $R \in \mathcal{M}_n(E_1, \ldots, E_n; H)$ such that

$$\Big\|\sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j)\Big\|_{\Sigma(\bar{F})} \leq \sigma \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + \Big\|\sum_{j=1}^{m} R(x_1^j, \ldots, x_n^j)\Big\|_H, \tag{3}$$

for any $m \in \mathbb{N}$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$.
On the other hand, if $\delta > \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))$ then, for certain Banach space G and certain operator $S \in \mathcal{M}_n(E_1, \ldots, E_n; G)$, it holds that

$$\Big\|\sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j)\Big\|_{\Delta(\bar{F})} \leq \delta \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + \Big\|\sum_{j=1}^{m} S(x_1^j, \ldots, x_n^j)\Big\|_G, \tag{4}$$

for all $m \in \mathbb{N}$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n, j = 1, \ldots, m$.
Let $\varepsilon > 0$. Define $V := (H \oplus G)_1$, and let $P \in \mathcal{L}(E_1, \ldots, E_n; V)$ be given by

$$P(x_1, \ldots, x_n) := (2 + \varepsilon)t\,(R(x_1, \ldots, x_n), S(x_1, \ldots, x_n)).$$

Due to $P = (2+\varepsilon)t(\psi_0 \circ R + \psi_1 \circ S)$, where $\psi_0 : H \to V$ and $\psi_1 : G \to V$ are the natural inclusions, it holds that $P \in \mathcal{M}_n(E_1, \ldots, E_n; V)$.

For any $m \in \mathbb{N}$ and $x_1^j \in E_1, \ldots, x_n^j \in E_n$, there exists a decomposition of $\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)$ as $\sum_{j=1}^m T(x_1^j, \ldots, x_n^j) = y_0 + y_1$, with $y_k \in F_k$ and

$$\|y_k\|_{F_k} \leq \|y_0\|_{F_0} + \|y_1\|_{F_1} \leq (1+\varepsilon)\Big\| \sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Sigma(\bar{F})}, \quad k = 0, 1. \quad (5)$$

It follows from (5) and (3) that, for $k = 0, 1$,

$$\|y_k\|_{F_k} \leq (1+\varepsilon)\sigma \sum_{j=1}^m \|x_1^j\| \cdots \|x_n^j\| + (1+\varepsilon)\Big\| \sum_{j=1}^m R(x_1^j, \ldots, x_n^j)\Big\|_H. \quad (6)$$

Since also $y_k \in \Delta(\bar{F})$, $k = 0, 1$, using (5) we obtain that

$$\begin{aligned}
\|y_k\|_{F_{1-k}} &\leq \Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{F_{1-k}} + \|y_{1-k}\|_{F_{1-k}} \\
&\leq \Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Delta(\bar{F})} + (1+\varepsilon)\Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Sigma(\bar{F})} \\
&\leq (2+\varepsilon)\Big\|\sum_{j=1}^m T(x_1^j, \ldots, x_n^j)\Big\|_{\Delta(\bar{F})}.
\end{aligned}$$

By (4), we have for $k = 0, 1$ that

$$\|y_k\|_{F_{1-k}} \leq (2+\varepsilon)\delta \sum_{j=1}^{m} \|x_1^j\| \ldots \|x_n^j\| + (2+\varepsilon)\Big\| \sum_{j=1}^{m} S(x_1^j, \ldots, x_n^j)\Big\|_G. \tag{7}$$

Taking into account (2), (6), (7) and the fact that $t \geq 1$, it holds that

$$\begin{aligned}
&\Big\| \sum_{j=1}^{m} T(x_1^j, \ldots, x_n^j)\Big\|_F \leq \|y_0\|_F + \|y_1\|_F \leq Ct^{\theta} J(t^{-1}, y_0) + Ct^{-\theta} J(t, y_1) \\
&\leq C\eta t \max\left\{\|y_0\|_{F_0}, t^{-1}\|y_0\|_{F_1}\right\} + C\eta \max\left\{\|y_1\|_{F_0}, t\|y_1\|_{F_1}\right\} \\
&\leq C\eta t \max\Big\{(1+\varepsilon)\sigma \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (1+\varepsilon)\Big\| \sum_{j=1}^{m} R(x_1^j, \ldots, x_n^j)\Big\|_H, \\
&\qquad t^{-1}\Big[(2+\varepsilon)\delta \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (2+\varepsilon)\Big\| \sum_{j=1}^{m} S(x_1^j, \ldots, x_n^j)\Big\|_G\Big]\Big\} \\
&\quad + C\eta \max\Big\{(2+\varepsilon)\delta \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (2+\varepsilon)\Big\| \sum_{j=1}^{m} S(x_1^j, \ldots, x_n^j)\Big\|_G, \\
&\qquad t\Big[(1+\varepsilon)\sigma \sum_{j=1}^{m} \|x_1^j\| \cdots \|x_n^j\| + (1+\varepsilon)\Big\| \sum_{j=1}^{m} R(x_1^j, \ldots, x_n^j)\Big\|_H\Big]\Big\}
\end{aligned}$$

Whence

$$\Big\| \sum_{j=1}^{m} T(x_1^j,\dots,x_n^j)\Big\|_F \leq$$

$$\leq 2C\eta \max\Big\{(1+\varepsilon)\sigma t \sum_{j=1}^{m} \|x_1^j\|\cdots\|x_n^j\| + (1+\varepsilon)t\Big\|\sum_{j=1}^{m} R(x_1^j,\dots,x_n^j)\Big\|_H,$$

$$(2+\varepsilon)\delta \sum_{j=1}^{m} \|x_1^j\|\cdots\|x_n^j\| + (2+\varepsilon)\Big\|\sum_{j=1}^{m} S(x_1^j,\dots,x_n^j)\Big\|_G\Big\}$$

$$\leq 2C\eta \max\{(1+\varepsilon)\sigma t,(2+\varepsilon)\delta\} \sum_{j=1}^{m} \|x_1^j\|\cdots\|x_n^j\|$$

$$+ (1+\varepsilon)t\Big\|\sum_{j=1}^{m} R(x_1^j,\dots,x_n^j)\Big\|_H + (2+\varepsilon)\Big\|\sum_{j=1}^{m} S(x_1^j,\dots,x_n^j)\Big\|_G$$

$$\leq 2C\eta \max\{(1+\varepsilon)\sigma t,(2+\varepsilon)\delta\} \sum_{j=1}^{m} \|x_1^j\|\cdots\|x_n^j\|$$

$$+ (2+\varepsilon)t\Big(\Big\|\sum_{j=1}^{m} R(x_1^j,\dots,x_n^j)\Big\|_H + \Big\|\sum_{j=1}^{m} S(x_1^j,\dots,x_n^j)\Big\|_G\Big)$$

$$= 2C\eta \max\{(1+\varepsilon)\sigma t,(2+\varepsilon)\delta\} \sum_{j=1}^{m} \|x_1^j\|\cdots\|x_n^j\| + \Big\|\sum_{j=1}^{m} P(x_1^j,\dots,x_n^j)\Big\|_V.$$

Thus,

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq 2C\eta \max\{(1+\varepsilon)\sigma t, (2+\varepsilon)\delta\}.$$

Therefore,

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq 2C\eta \max\{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))t, 2\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))\}. \tag{8}$$

We consider the following cases:

i) If $\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) = 0$, then $\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) = 0$ as well, since η is arbitrary.

ii) Suppose that $\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) > 0$. Then, $\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) > 0$ because

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F})) \leq \beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})).$$

Take

$$\eta := \max\left\{ \left(\frac{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))}\right)^{\theta}, \left(\frac{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))}\right)^{1-\theta} \right\}.$$

The real number

$$t := \frac{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))}{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))} \geq 1$$

satisfies (2). If we denote $\Theta := \min\{\theta, 1-\theta\}$ and substitute these concrete choices of η and t in (8), we obtain that

$$\begin{aligned} &\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq \\ &\quad \leq 4C\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F})) \left(\frac{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))}{\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))}\right)^{\Theta} \\ &\quad = 4C\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta}\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta}. \end{aligned}$$

□

Theorem 8

Let $\mathcal{M}_n$ be an n-ideal, let $E_1, \dots, E_n$ be Banach spaces and let $\bar{F} = (F_0, F_1)$ be a Banach couple. Assume that F is of class $\mathcal{C}_J(\theta, \bar{F})$ with constant C. For $T \in \mathcal{L}(E_1, \dots, E_n; \Delta(\bar{F}))$,

$$\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to F) \leq$$
$$4C\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Delta(\bar{F}))^{1-\Theta}\,\beta_{\mathcal{M}_n}(T : E_1 \times \cdots \times E_n \to \Sigma(\bar{F}))^{\Theta},$$

where $\Theta = \min\{\theta, 1-\theta\}$.

As a straightforward consequence of Theorem 8 we deduce the next result.

Corollary 9

Let $\mathcal{M}_n$ be a closed injective n-ideal. Let $\bar{F} = (F_0, F_1)$ be a Banach couple and let F be of class $\mathcal{C}_J(\theta, \bar{F})$. For $T \in \mathcal{L}(E_1, \dots, E_n; \Delta(\bar{F}))$, it follows that

$$T \in \mathcal{M}_n(E_1, \dots, E_n; F) \text{ if and only if } T \in \mathcal{M}_n(E_1, \dots, E_n; \Sigma(\bar{F})).$$

Corollary 9 can be applied to $\mathcal{M}_n = [\mathcal{I}_1, \dots, \mathcal{I}_n]$, with $\mathcal{I}_1, \dots, \mathcal{I}_n$ closed injective operator ideals. **If $\mathcal{I}_1 = \cdots = \mathcal{I}_n = \mathcal{I}$ and $\mathcal{I}$ is any of the following closed injective ideals**: **compact operators**, **weakly compact operators**, **strictly singular operators**, **Rosenthal operators**, **Banach-Saks operators**, or **Asplund operators**, **we get an extension to the multilinear case of known interpolation results** on such ideals.

Combining Corollaries 7 and 9, immediately follows:

Corollary 10

Let $\mathcal{M}_n$ be a closed surjective injective n-ideal. Let $\bar{E}_j, j = 1, \ldots n$, and $\bar{F}$ be Banach couples. Assume that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j), j = 1, \ldots, n$, and F is of class $\mathcal{C}_J(\eta, \bar{F})$. For every $T \in \mathcal{L}(\Sigma(\bar{E}_1), \ldots, \Sigma(\bar{E}_n); \Delta(\bar{F}))$, it holds that

$$T \in \mathcal{M}_n(E_1, \ldots, E_n; F) \text{ if and only if } T \in \mathcal{M}_n(\Delta(\bar{E}_1), \ldots, \Delta(\bar{E}_n); \Sigma(\bar{F})).$$

As another application we get interpolation results for **compact multilinear operators** and **weakly compact multilinear operators**.

- $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called compact (resp. weakly compact) when $T(B_{E_1} \times \cdots \times B_{E_n})$ is relatively compact (resp. relatively weakly compact) in F.

Corollary 11

Let $\bar{E}_j, j = 1, \ldots n$, and $\bar{F}$ be Banach couples. Assume that E_j is of class $\mathcal{C}_K(\theta_j, \bar{E}_j), j = 1, \ldots, n$, and F is of class $\mathcal{C}_J(\eta, \bar{F})$. It holds that, for every $T : \Sigma(\bar{E}_1) \times \cdots \times \Sigma(\bar{E}_n) \to \Delta(\bar{F})$,

(a) $T : E_1 \times \cdots \times E_n \to F$ is compact if and only if the restriction $T : \Delta(\bar{E}_1) \times \cdots \times \Delta(\bar{E}_n) \to \Sigma(\bar{F})$ is compact.

(b) $T : E_1 \times \cdots \times E_n \to F$ is weakly compact if and only if the restriction $T : \Delta(\bar{E}_1) \times \cdots \times \Delta(\bar{E}_n) \to \Sigma(\bar{F})$ is weakly compact.

Thank you so much

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