Descriptive complexity of diameter 2 properties

Esteban Martínez Vañó

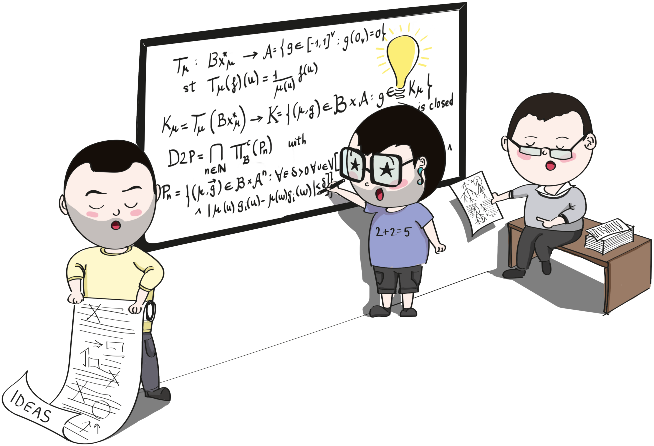
Universidad de Granada
Departamento de Análisis Matemático

New perspectives in Banach spaces and Banach lattices
July 2024, Castro Urdiales (Spain)

UNIVERSIDAD
DE GRANADA

My research is funded by:

- Grant PRE2022-101438 funded by MICIU/AEI/10.13039/501100011033 and by "ESF+".
- Grant PID2021-122126NB-C31 funded by MCIU/AEI/FEDER/UE.
- Grant FQM-0185 funded by Junta de Andalucía .


with
is closed
2+2=5
IDEAS

Bossard's idea ([Bos93], [Bos02])

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Bossard's idea ([Bos93], [Bos02])

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

Bossard's idea ([Bos93], [Bos02])

Analytic set: Continuous image of Polish space
Coanalytic set: Complement of analytic set
Borel = Analytic $\cap$ Coanalytic

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

Bossard's idea ([Bos93], [Bos02])

Analytic set: Continuous image of Polish space
Coanalytic set: Complement of analytic set
Borel = Analytic ∩ Coanalytic

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

- RNP is complete coanalytic.

Bossard's idea ([Bos93], [Bos02])

Analytic set: Continuous image of Polish space
Coanalytic set: Complement of analytic set
Borel = Analytic ∩ Coanalytic

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

- RNP is complete coanalytic.
- Uniformly convex spaces are Borel.

Bossard's idea ([Bos93], [Bos02])

Analytic set: Continuous image of Polish space
Coanalytic set: Complement of analytic set
Borel = Analytic $\cap$ Coanalytic

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

- RNP is complete coanalytic.
- Uniformly convex spaces are Borel.

Problem:

Bossard's idea ([Bos93], [Bos02])

Analytic set: Continuous image of Polish space
Coanalytic set: Complement of analytic set
Borel = Analytic $\cap$ Coanalytic

$\underbrace{\mathrm{SB}(C(\Delta))}_{\text{Closed subspaces}}$ endowed with the Effros-Borel structure.

Allows to distinguish between Borel and non-Borel classes:

- RNP is complete coanalytic.
- Uniformly convex spaces are Borel.

Problem:

No canonical topology $\Rightarrow$ No distinction between Borel classes

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

$$V = \{v \in \mathbb{Q}^{\mathbb{N}} : v \text{ has finite support}\}$$
$$\mathcal{P} = \{\mu \in \mathbb{R}^{V} : \mu \text{ is a seminorm}\}$$

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

$$V = \{v \in \mathbb{Q}^{\mathbb{N}} : v \text{ has finite support}\}$$
$$\mathcal{P} = \{\mu \in \mathbb{R}^{V} : \mu \text{ is a seminorm}\}$$

  - Extend μ to c_{00}: $\bar{\mu}$

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

$$V = \{v \in \mathbb{Q}^{\mathbb{N}} : v \text{ has finite support}\}$$
$$\mathcal{P} = \{\mu \in \mathbb{R}^{V} : \mu \text{ is a seminorm}\}$$

  - Extend μ to c_{00}: $\bar{\mu}$
  - Pass to the quotient with $N_\mu = \{x \in c_{00} : \bar{\mu}(x) = 0\}$: $\tilde{\mu}$

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

$$V = \{v \in \mathbb{Q}^{\mathbb{N}} : v \text{ has finite support}\}$$
$$\mathcal{P} = \{\mu \in \mathbb{R}^{V} : \mu \text{ is a seminorm}\}$$

  - Extend μ to c_{00}: $\bar{\mu}$
  - Pass to the quotient with $N_\mu = \{x \in c_{00} : \bar{\mu}(x) = 0\}$: $\tilde{\mu}$
  - Complete $(c_{00}/N_\mu, \tilde{\mu})$: $(X_\mu, \hat{\mu})$

Extensions of Bossard's idea

- Godefroy and Saint-Raymond [GSR18]: Admissible topologies.
- Cúth, Doležal, Doucha and Kurka [CDDK22b], [CDDK22a]:

$$V = \{v \in \mathbb{Q}^{\mathbb{N}} : v \text{ has finite support}\}$$
$$\mathcal{P} = \{\mu \in \mathbb{R}^{V} : \mu \text{ is a seminorm}\}$$

- Extend μ to c_{00}: $\bar{\mu}$
- Pass to the quotient with $N_\mu = \{x \in c_{00} : \bar{\mu}(x) = 0\}$: $\tilde{\mu}$
- Complete $(c_{00}/N_\mu, \tilde{\mu})$: $(X_\mu, \hat{\mu})$

$(X_\mu, \hat{\mu})$ is a separable Banach space, V countable and dense.

Extensions of Bossard's idea

Given a separable Banach space $(X, \|\cdot\|)$ with a dense sequence $\{x_n\}$, we can obtain a seminorm $\mu \in \mathcal{P}$ such that $X \equiv X_\mu$ by defining for every $v = \sum a_n e_n \in V$ (with $\{e_n\}$ the canonical basis of c_{00})

$$\mu(v) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\|.$$

Extensions of Bossard's idea

Given a separable Banach space $(X, \|\cdot\|)$ with a dense sequence $\{x_n\}$, we can obtain a seminorm $\mu \in \mathcal{P}$ such that $X \equiv X_\mu$ by defining for every $v = \sum a_n e_n \in V$ (with $\{e_n\}$ the canonical basis of c_{00})

$$\mu(v) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\| .$$

$$\mathcal{P}_\infty = \{\mu \in \mathcal{P} : X_\mu \text{ is infinite-dimensional}\}$$
$$\mathcal{B} = \{\mu \in \mathcal{P}_\infty : \bar{\mu} \text{ is a norm}\}$$

Diameter 2 properties

Recall: RNP is complete coanalytic.

Diameter 2 properties

Recall: RNP is complete coanalytic.

Diameter 2 properties

A Banach space X has:

Diameter 2 properties

Recall: RNP is complete coanalytic.

Diameter 2 properties

A Banach space X has:

- LD2P: Every slice of B_X has diameter 2.

Diameter 2 properties

Recall: RNP is complete coanalytic.

Diameter 2 properties

A Banach space X has:

- LD2P: Every slice of B_X has diameter 2.
- D2P: Every w-open set of B_X has diameter 2.

Complexity of isometry classes

What about the isometry classes?

Complexity of isometry classes

What about the isometry classes?

1. Give a purely geometrical characterization of the properties.

Complexity of isometry classes

What about the isometry classes?

1. Give a purely geometrical characterization of the properties.
   X has LD2P iff for every $\varepsilon > 0$

$$B_X = \overline{co}\left\{\frac{x+y}{2} : x, y \in B_X \wedge \|x-y\| \geq 2-\varepsilon\right\}$$

Complexity of isometry classes

What about the isometry classes?

1. Give a purely geometrical characterization of the properties.
   X has LD2P iff for every $\varepsilon > 0$

$$B_X = \overline{co}\left\{\frac{x+y}{2} : x, y \in B_X \wedge \|x-y\| \geq 2-\varepsilon\right\}$$

2. Find a way to “talk about the dual space”.

Codification of the dual space in $\mathcal{B}$

Given $\mu \in \mathcal{B}$ we define the map

$$\begin{aligned} T_\mu : B_{X_\mu^*} &\longrightarrow \quad [-1,1]^V \\ f &\longmapsto T_\mu(f) : V \longrightarrow [-1,1] \\ & \qquad\qquad\qquad u \longmapsto \begin{cases} 0 & \text{if} \quad u = 0_V \\ \dfrac{1}{\mu(u)} f(u) & \text{if} \quad u \neq 0_V \end{cases} \end{aligned}$$

Codification of the dual space in $\mathcal{B}$

Given $\mu \in \mathcal{B}$ we define the map

$$\begin{aligned} T_\mu : B_{X_\mu^*} &\longrightarrow \; [-1,1]^V \\ f &\longmapsto T_\mu(f) : V \longrightarrow [-1,1] \\ & \qquad\qquad u \longmapsto \begin{cases} 0 & \text{if} \quad u = 0_V \\ \dfrac{1}{\mu(u)} f(u) & \text{if} \quad u \neq 0_V \end{cases} \end{aligned}$$

We will denote the image of this map by K_μ. Observe then that given $g \in [-1,1]^V$ we have that

$$g \in K_\mu \leftrightarrow g(0_V) = 0 \wedge \exists f \in B_{X_\mu^*} \forall u \in V \, f(u) = \mu(u) g(u).$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Let's denote the isometry class of the spaces with the D2P as

$$\widehat{\text{D2P}} = \{\mu \in \mathcal{B} : X_\mu \text{ has the D2P}\}$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Let's denote the isometry class of the spaces with the D2P as

$$\widehat{\text{D2P}} = \{\mu \in \mathcal{B} : X_\mu \text{ has the D2P}\}$$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall \varepsilon > 0\, \forall \delta > 0\, \forall u \in V\,[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V\,(\mu(v-w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\}\,[|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta])]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$
- $\alpha \to \beta \equiv \neg\alpha \vee \beta$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$
- $\alpha \to \beta \equiv \neg\alpha \vee \beta$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall\varepsilon > 0\, \forall\delta > 0\, \forall u \in V\, [\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V\, (\mu(v - w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\}\, [|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta])]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$
- $\alpha \to \beta \equiv \neg\alpha \vee \beta$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall\varepsilon > 0\, \forall\delta > 0\, \forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V \big(\mu(v - w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta\big]\big)\big]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$
- $\alpha \to \beta \equiv \neg\alpha \vee \beta$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall\varepsilon > 0\, \forall\delta > 0\, \forall u \in V\, [\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V\, (\mu(v - w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\}\, [|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta])]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Recall:

- $U(u; f_1, \cdots, f_n; \delta) = \{x \in B_X : |f_i(u - x)| < \delta\}$
- $\alpha \to \beta \equiv \neg\alpha \vee \beta$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall \varepsilon > 0 \, \forall \delta > 0 \, \forall u \in V \, [\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\} \, g_i \notin K_\mu \vee \exists v, w \in V \, (\mu(v - w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \, [|\mu(u) g_i(u) - \mu(v) g_i(v)| < \delta \wedge |\mu(u) g_i(u) - \mu(w) g_i(w)| < \delta])]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

We have that

$$\mu \in \widehat{\text{D2P}} \leftrightarrow \forall n \in \mathbb{N}\, \forall \vec{g} \in \left([-1,1]^V\right)^n (\mu, \vec{g}) \in P_n$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

We have that

$$\mu \in \widehat{\mathrm{D2P}} \leftrightarrow \forall n \in \mathbb{N}\, \forall \vec{g} \in \left([-1,1]^V\right)^n (\mu, \vec{g}) \in P_n$$

The last formula is equivalent to

$$\widehat{\mathrm{D2P}} = \bigcap_{n=1}^{\infty} \pi^c_{\mathcal{B}}(P_n)$$

where

$$\pi^c_{\mathcal{B}}(P_n) = \left(\pi_{\mathcal{B}}\left(P_n^c\right)\right)^c$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall \varepsilon > 0\, \forall \delta > 0\, \forall u \in V\,\big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V\,(\mu(v-w) > 2-\varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\}\,[|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta])\big]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall m \in \mathbb{N}\, \forall k \in \mathbb{N}\, \forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V \big(\mu(v-w) > 2 - 1/m \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < 1/k \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < 1/k\big]\big)\big]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$\begin{aligned}
P_n =&\Big\{(\mu,\vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall m \in \mathbb{N}\, \forall k \in \mathbb{N}\, \forall u \in V \big[\mu(u) > 1 \vee \\
&\exists i \in \{1,\cdots,n\}\, g_i \notin K_\mu \vee \exists v,w \in V\big(\mu(v-w) > 2 - 1/m \wedge \\
&\mu(v),\mu(w) < 1 \wedge \forall i \in \{1,\cdots,n\}\big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < 1/k \\
&\wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < 1/k\big]\big)\big]\Big\}.
\end{aligned}$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall m \in \mathbb{N}\, \forall k \in \mathbb{N}\, \forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V \big(\mu(v-w) > 2 - 1/m \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < 1/k \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < 1/k\big]\big)\big]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$P_n = \Big\{(\mu,\vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall m \in \mathbb{N}\, \forall k \in \mathbb{N}\, \forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1,\cdots,n\}\, g_i \notin K_\mu \vee \exists v,w \in V \big(\mu(v-w) > 2 - 1/m \wedge \mu(v),\mu(w) < 1 \wedge \forall i \in \{1,\cdots,n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < 1/k \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < 1/k\big]\big)\big]\Big\}.$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall m \in \mathbb{N} \,\forall k \in \mathbb{N} \,\forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\}\, g_i \notin K_\mu \vee \exists v, w \in V \big(\mu(v - w) > 2 - 1/m \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < 1/k \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < 1/k\big]\big)\big]\Big\}.$$

In general, the set

$$K = \Big\{(\mu, g) \in \mathcal{B} \times [-1,1]^V : g \in K_\mu\Big\}$$

is closed.

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

- UC_∞ is dense in $\mathcal{B}$.

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

- UC_∞ is dense in $\mathcal{B}$.
- $\mathrm{UC}_\infty \subset \mathcal{B} \setminus \widehat{\mathrm{D2P}}$.

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

- UC_∞ is dense in $\mathcal{B}$.
- $\mathrm{UC}_\infty \subset \mathcal{B} \setminus \widehat{\mathrm{D2P}}$.
- $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ is dense in $\mathcal{B}$.

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

- UC_∞ is dense in $\mathcal{B}$.
- $\mathrm{UC}_\infty \subset \mathcal{B} \setminus \widehat{\mathrm{D2P}}$.
- $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ is dense in $\mathcal{B}$.
- In [CDDK22a] it is proven that the isometry class of the Gurariĭ space is G_δ dense in $\mathcal{B}$.

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

- UC_∞ is dense in $\mathcal{B}$.
- $\mathrm{UC}_\infty \subset \mathcal{B} \setminus \widehat{\mathrm{D2P}}$.
- $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ is dense in $\mathcal{B}$.
- In [CDDK22a] it is proven that the isometry class of the Gurariĭ space is G_δ dense in $\mathcal{B}$.

If $\widehat{\mathrm{D2P}}$ is F_σ in $\mathcal{B}$, then $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ is G_δ and dense. By Baire's theorem $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ has to intersect the isometry class of the Gurariĭ space, but this is a contradiction because the Gurariĭ space has de D2P.

Compendium of results

	LD2P	D2P	SD2P
$\mathcal{B}$	G_δ-complete	G_δ-complete	G_δ-complete
$\mathcal{P}_\infty$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete

	DLD2P	DD2P	DP
$\mathcal{B}$	G_δ-complete	G_δ-complete	G_δ-complete
$\mathcal{P}_\infty$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete

	LOH	WOH	OH
$\mathcal{B}$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete
$\mathcal{P}_\infty$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete

Benoît Bossard.
Codages des espaces de Banach séparables. Familles analytiques ou coanalytiques d'espaces de Banach.
Comptes rendus de l'Académie des sciences. Série 1, Mathématique, 316(10):1005–1010, 1993.

Benoît Bossard.
A coding of separable banach spaces. analytic and coanalytic families of banach spaces.
Fundamenta Mathematicae, 2(172):117–152, 2002.

Marek Cúth, Martin Doležal, Michal Doucha, and Ondřej Kurka.
Polish spaces of Banach spaces: Complexity of isometry and isomorphism classes.
Journal of the Institute of Mathematics of Jussieu, pages 1–39, 2022.

Marek Cúth, Martin Doležal, Michal Doucha, and Ondřej Kurka.
Polish spaces of Banach spaces.
Forum of Mathematics, Sigma, 10:e26, 2022.

Gilles Godefroy and Jean Saint-Raymond.
Descriptive complexity of some isomorphism classes of Banach spaces.
Journal of Functional Analysis, 275(4):1008–1022, 2018.

Ginés López-Pérez, Esteban Martínez Vañó, and Abraham Rueda Zoca.
Computing Borel complexity of some geometrical properties in Banach spaces.
arXiv preprint arXiv:2404.19457, 2024.

Thank you!