Lipschitz free spaces isomorphic to ℓ_1

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Joint work with
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M will be a finite metric space with distance d.

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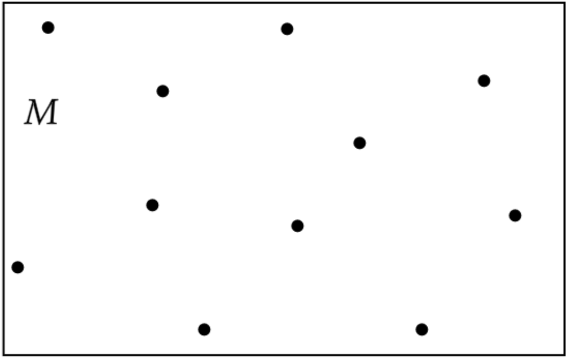
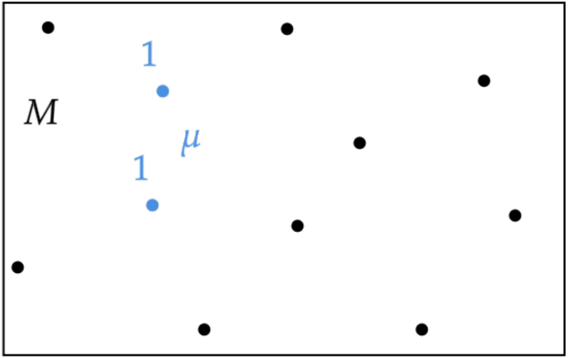
$\mathcal{F}(M)$ will be the Lipschitz free space or Transportation cost space over M.

Transportation cost space - Definition



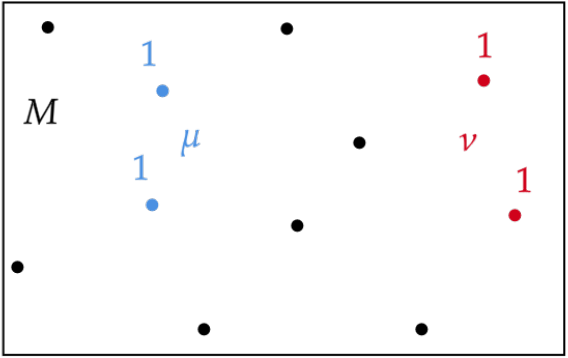
Transportation cost space - Definition

$\mu \in \mathcal{P}(M)$ positive measure over M



Transportation cost space - Definition

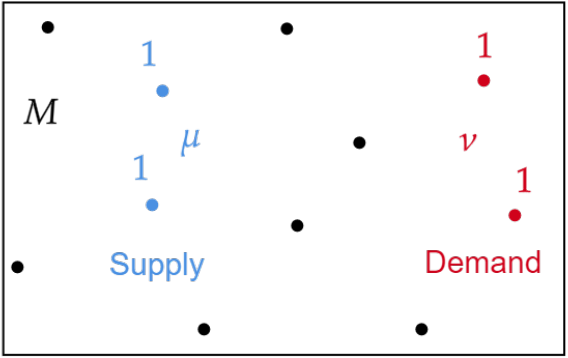
$\mu, \nu \in \mathcal{P}(M)$ positive measures over M



Transportation cost space - Definition

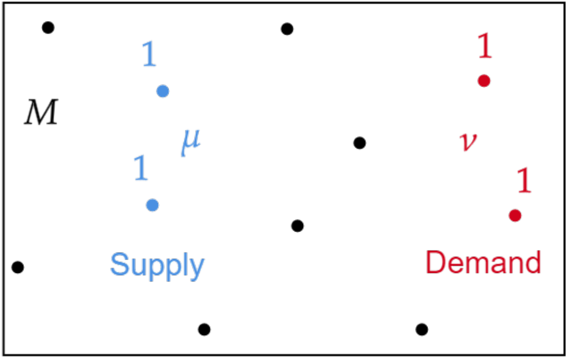
$\mu, \nu \in \mathcal{P}(M)$ positive measures over M

Same mass, disjoint support



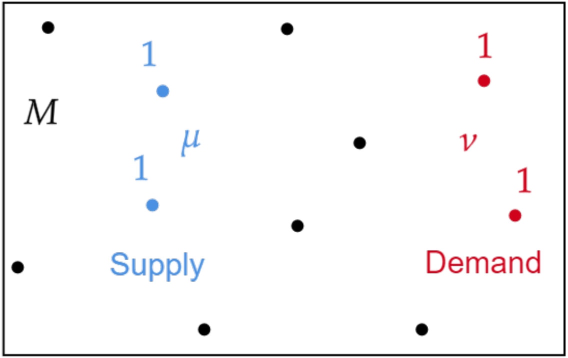
Transportation cost space - Definition

$$\mu = \text{Supply} \quad , \quad \nu = \text{Demand}$$



Transportation cost space - Definition

$$\mu = \text{Supply} \quad , \quad \nu = \text{Demand} \quad , \quad \mu - \nu \in \mathcal{F}(M)$$



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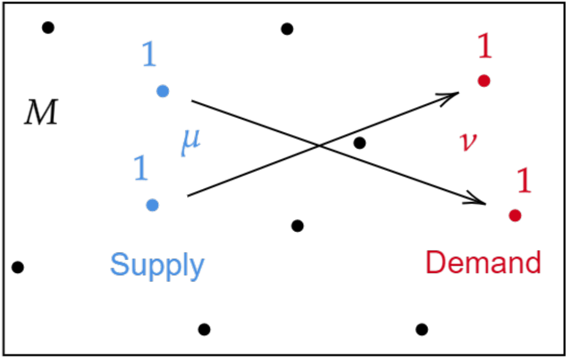

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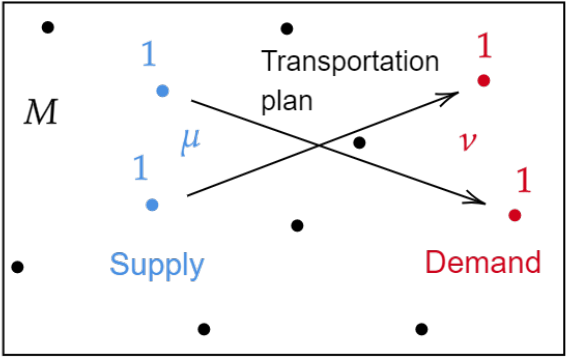
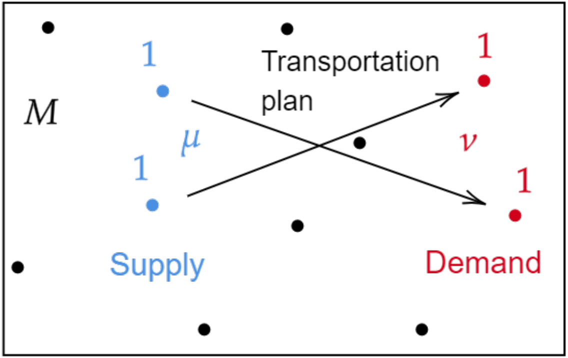


Transportation cost space - Definition

$$\mu = \text{Supply} \quad , \quad \nu = \text{Demand} \quad , \quad \mu - \nu \in \mathcal{F}(M)$$

Cost of Transportation plan$=d_1 + d_2$



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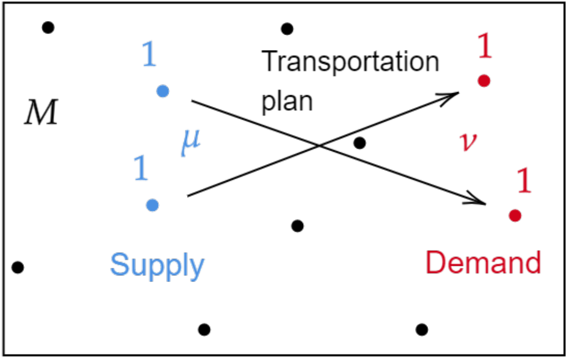
Cost of Transportation plan$= d_1 + d_2$

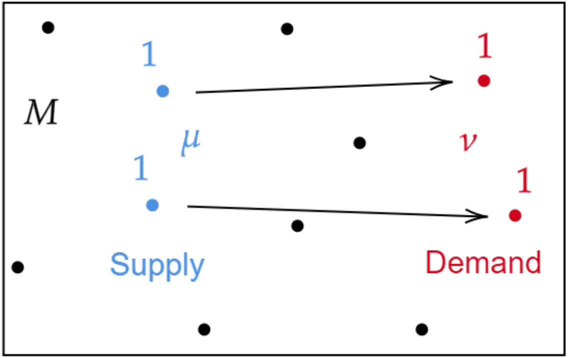
Not optimal!



Transportation cost space - Definition

$$\mu = \text{Supply} \quad , \quad \nu = \text{Demand} \quad , \quad \mu - \nu \in \mathcal{F}(M)$$

$$\text{Cost of Transportation plan} = d_1 + d_2$$

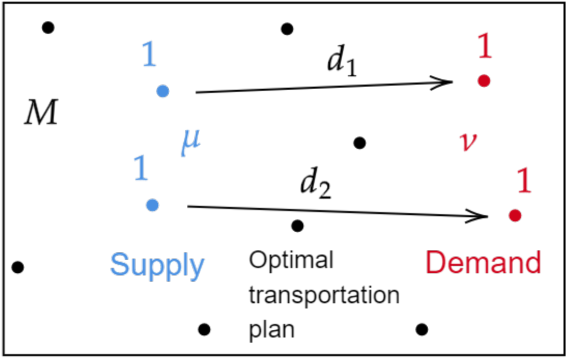


Transportation cost space - Definition

$$\mu = \text{Supply} \quad , \quad \nu = \text{Demand} \quad , \quad \mu - \nu \in \mathcal{F}(M)$$

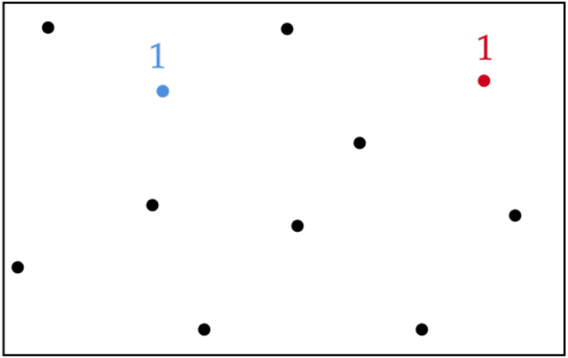
$$\text{Cost of Transportation plan} = d_1 + d_2$$

$$\textit{Optimal} \quad \Rightarrow \quad \|\mu - \nu\| = \textit{Cost} = d_1 + d_2$$



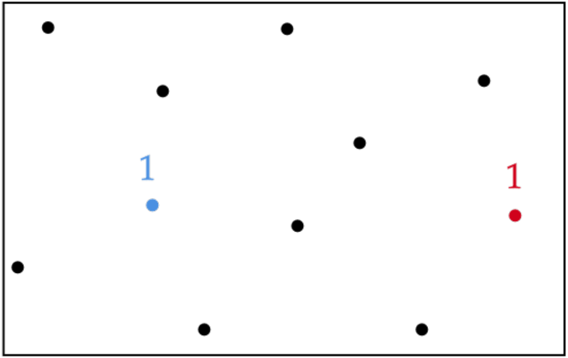
Transportation cost space - Linearity

$$\mu_1 - \nu_1 \in \mathcal{F}(M)$$



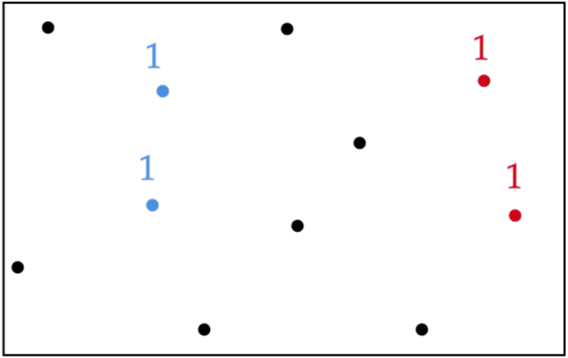
Transportation cost space - Linearity

$$\mu_2 - \nu_2 \in \mathcal{F}(M)$$



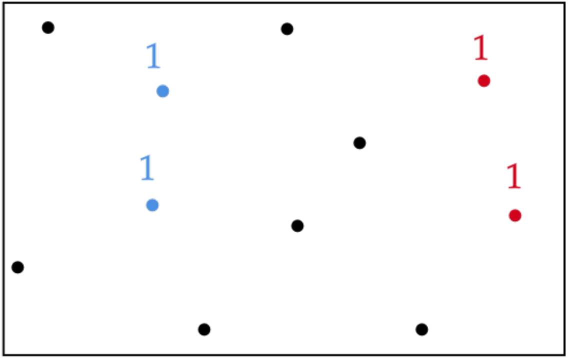
Transportation cost space - Linearity

$$(\mu_1 - \nu_1) + (\mu_2 - \nu_2)$$



Transportation cost space - Linearity

$$\underbrace{(\mu_1 + \mu_2)}_{\mu} - \underbrace{(\nu_1 + \nu_2)}_{\nu}$$



Transportation cost space - Linearity

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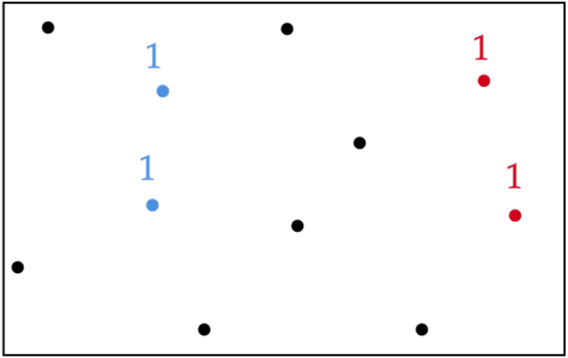
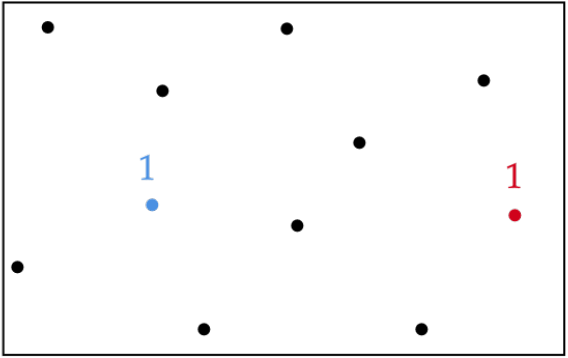


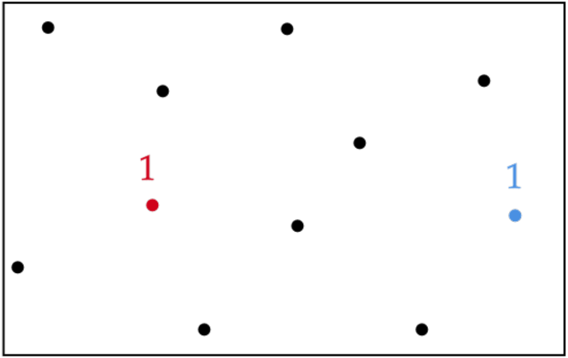
Transportation cost space - Linearity

$$\mu_2 - \nu_2$$



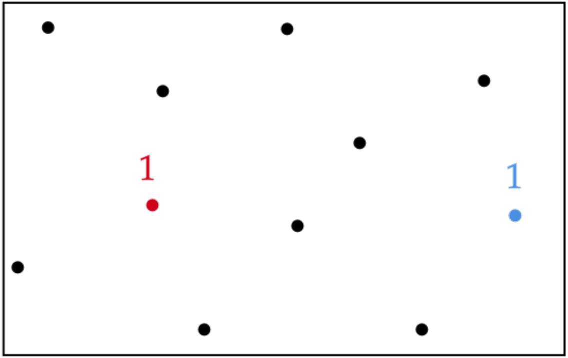
Transportation cost space - Linearity

$$(-1)(\mu_2 - \nu_2)$$



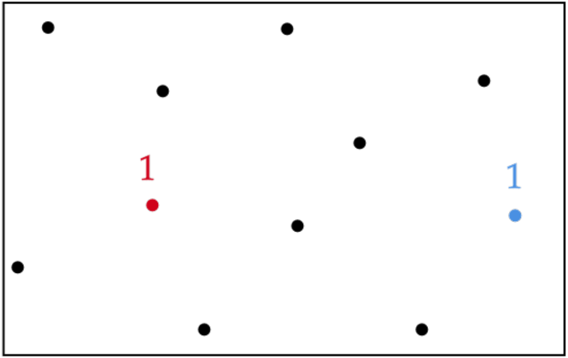
Transportation cost space - Linearity

$$\nu_2 - \mu_2$$



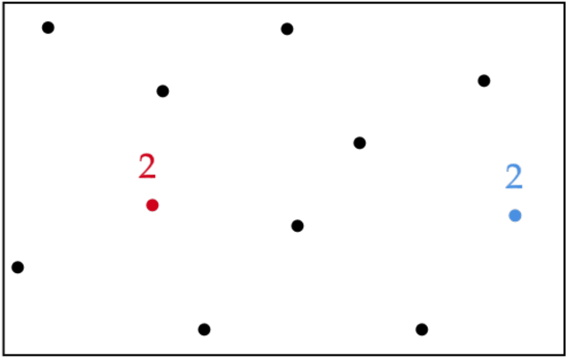
Transportation cost space - Linearity

$$\nu_2 - \mu_2 \in \mathcal{F}(M)$$



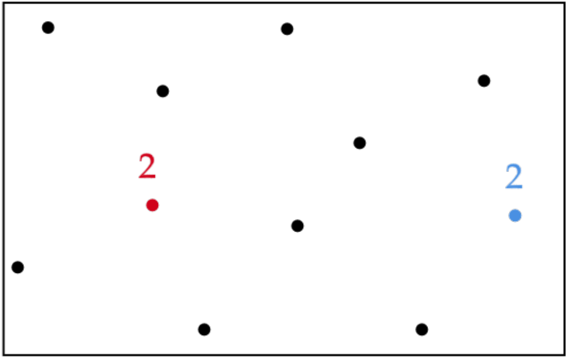
Transportation cost space - Linearity

$$2(\nu_2 - \mu_2)$$



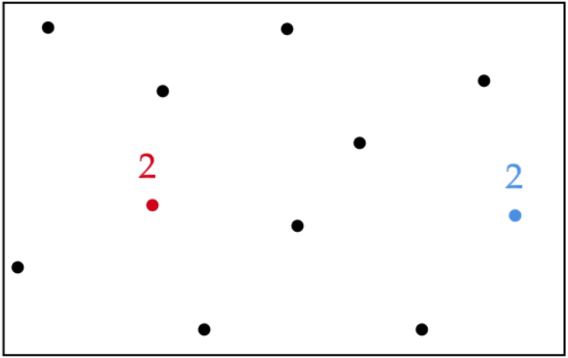
Transportation cost space - Linearity

$$2\nu_2 - 2\mu_2$$



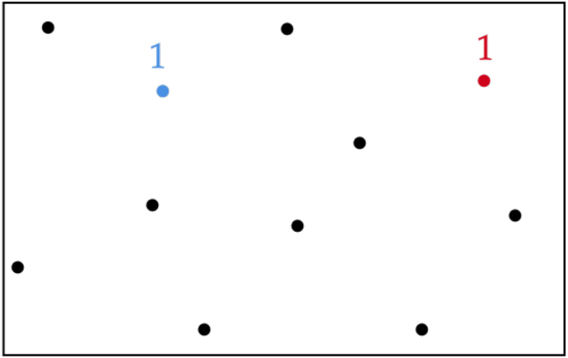
Transportation cost space - Linearity

$$2\nu_2 - 2\mu_2 \in \mathcal{F}(M)$$



Transportation cost space - ℓ_1 behaviour

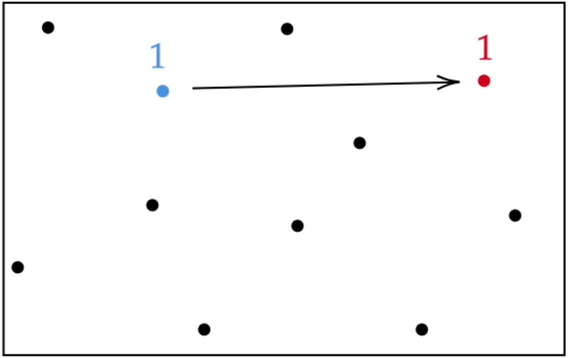
$$\mu_1 - \nu_1$$



Transportation cost space - ℓ_1 behaviour

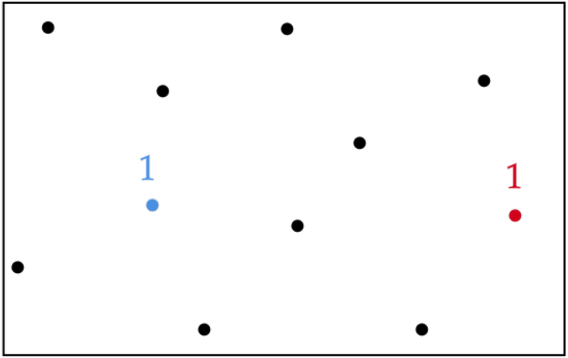
$$\mu_1 - \nu_1$$

$$\|\mu_1 - \nu_1\| = d_1$$



Transportation cost space - ℓ_1 behaviour

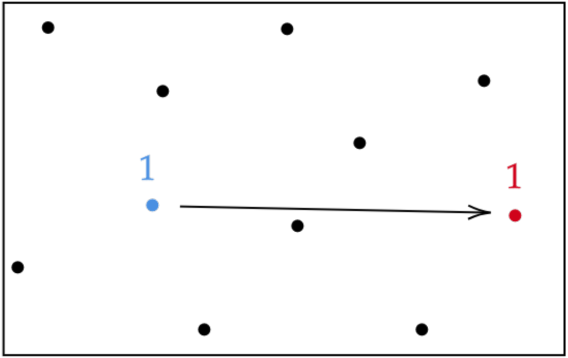
$$\mu_2 - \nu_2$$



Transportation cost space - ℓ_1 behaviour

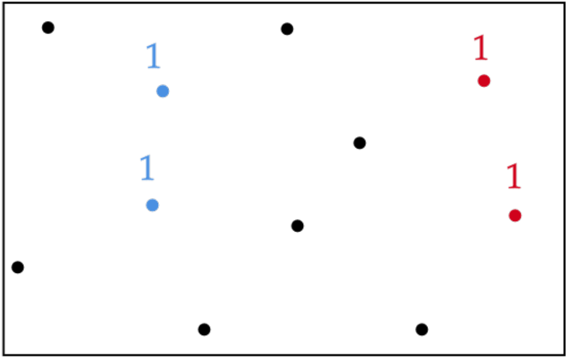
$$\mu_2 - \nu_2$$

$$\|\mu_2 - \nu_2\| = d_2$$



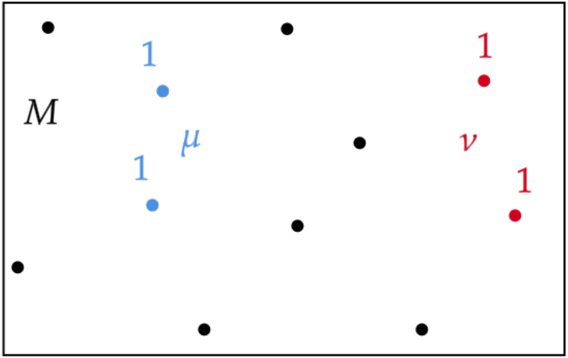
Transportation cost space - ℓ_1 behaviour

$$(\mu_1 - \nu_1) + (\mu_2 - \nu_2)$$



Transportation cost space - ℓ_1 behaviour

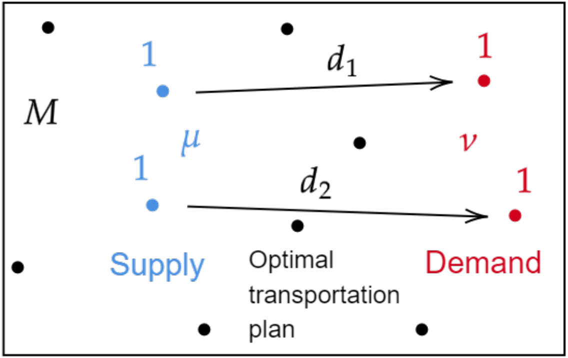
$$(\mu_1 - \nu_1) + (\mu_2 - \nu_2) = \mu - \nu$$



Transportation cost space - ℓ_1 behaviour

$$(\mu_1 - \nu_1) + (\mu_2 - \nu_2) = \mu - \nu$$

$$\|\mu - \nu\| = d_1 + d_2 = \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$$

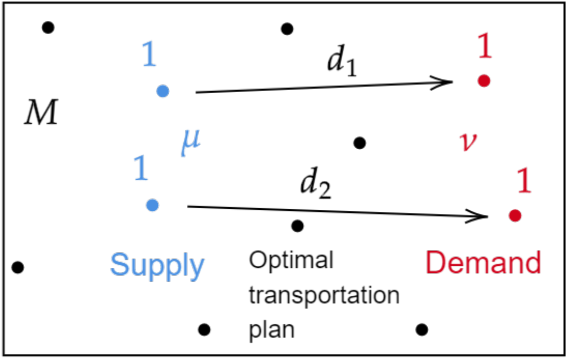


Transportation cost space - ℓ_1 behaviour

$$(\mu_1 - \nu_1) + (\mu_2 - \nu_2) = \mu - \nu$$

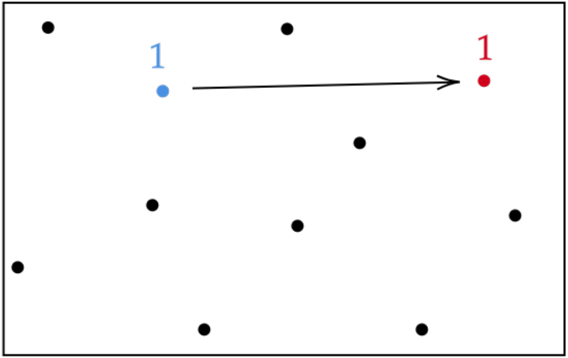
$$\|\mu - \nu\| = d_1 + d_2 = \|\mu_1 - \nu_1\| + \|\mu_2 - \nu_2\|$$

Sum of optimal transports is again optimal



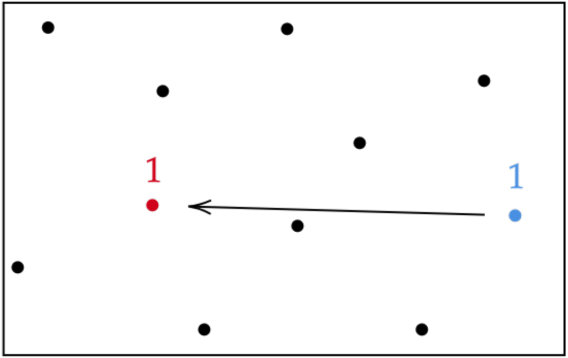
Transportation cost space - ℓ_1 behaviour

$$\mu_1 - \nu_1 \quad , \quad \|\mu_1 - \nu_1\| = d_1$$



Transportation cost space - ℓ_1 behaviour

$$\nu_2 - \mu_2 \quad , \quad \|\nu_2 - \mu_2\| = d_2$$

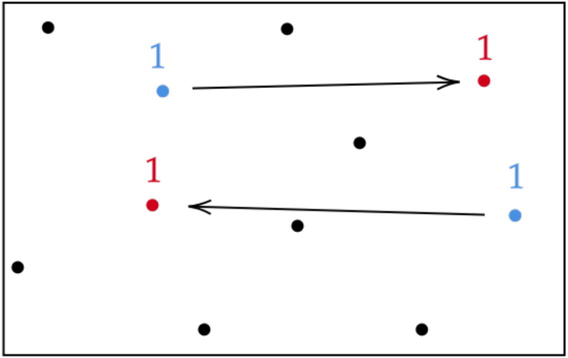


Transportation cost space - ℓ_1 behaviour

$$\mu_1 - \nu_1 \quad , \quad \|\mu_1 - \nu_1\| = d_1$$

$$\nu_2 - \mu_2 \quad , \quad \|\nu_2 - \mu_2\| = d_2$$

$$(\mu_1 - \nu_1) + (\nu_2 - \mu_2)$$

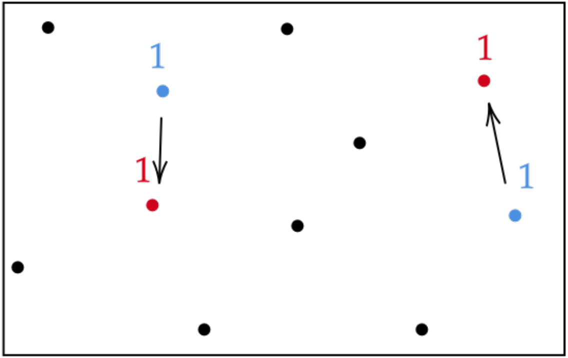


Transportation cost space - ℓ_1 behaviour

$$\mu_1 - \nu_1 \quad , \quad \|\mu_1 - \nu_1\| = d_1$$

$$\nu_2 - \mu_2 \quad , \quad \|\nu_2 - \mu_2\| = d_2$$

$$\|(\mu_1 - \nu_1) + (\nu_2 - \mu_2)\| < d_1 + d_2$$



ℓ_1 isometric Transportation cost space

$$M = \{1, \ldots, N\} \subset \mathbb{R}$$

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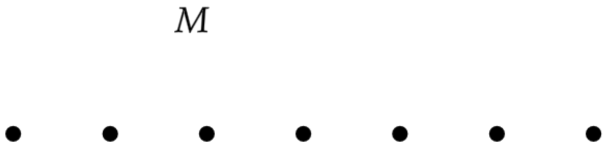
We need to find the ℓ_1 basis $(e_n)_{n=1}^N$



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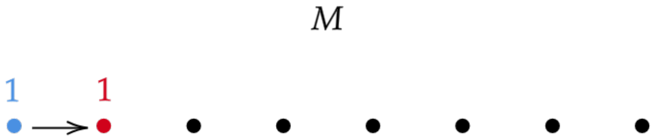
e_1



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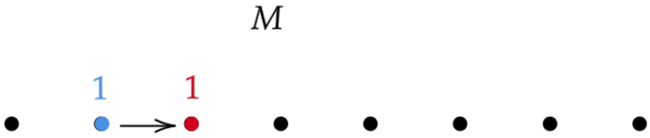
e_2



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e_3

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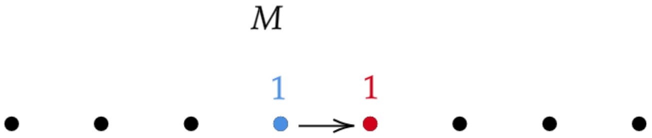
We need to find the ℓ_1 basis $(e_n)_{n=1}^N$

e_4



ℓ_1 isometric Transportation cost space

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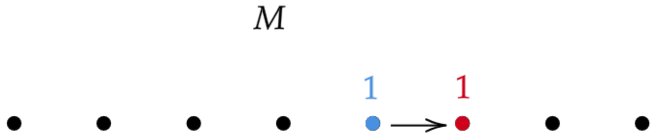
We need to find the ℓ_1 basis $(e_n)_{n=1}^N$

e_5



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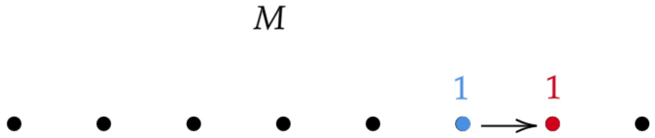
$$M = \{1, \dots, N\} \subset \mathbb{R}$$

We need to find the ℓ_1 basis $(e_n)_{n=1}^N$

$$e_6$$



ℓ_1 isometric Transportation cost space

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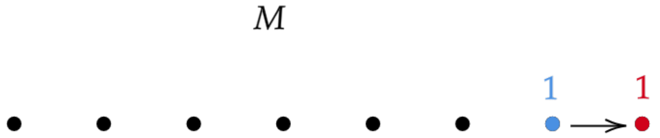
We need to find the ℓ_1 basis $(e_n)_{n=1}^N$

e_7



ℓ_1 isometric Transportation cost space

$$M = \{1, \ldots, N\} \subset \mathbb{R}$$

$$\mu - \nu \in \mathcal{F}(M)$$

ℓ_1 isometric Transportation cost space

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$$\mu - \nu \in \mathcal{F}(M)$$

$$\|\mu - \nu\| =$$

ℓ_1 isometric Transportation cost space

$$M = \{1, \ldots, N\} \subset \mathbb{R}$$

$$\mu - \nu \in \mathcal{F}(M)$$

$$\|\mu - \nu\| = \|e_2\|$$

ℓ_1 isometric Transportation cost space

$$M = \{1, \dots, N\} \subset \mathbb{R}$$

$$\mu - \nu \in \mathcal{F}(M)$$

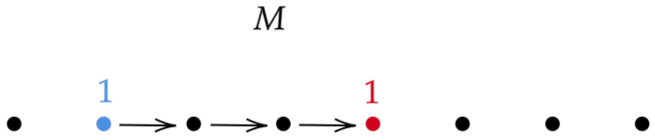
$$\|\mu - \nu\| = \|e_2\| + \|e_3\|$$

ℓ_1 isometric Transportation cost space

$$M = \{1, \ldots, N\} \subset \mathbb{R}$$

$$\mu - \nu \in \mathcal{F}(M)$$

$$\|\mu - \nu\| = \|e_2\| + \|e_3\| + \|e_4\|$$



ℓ_1 isometric Transportation cost space

This works because M is a tree

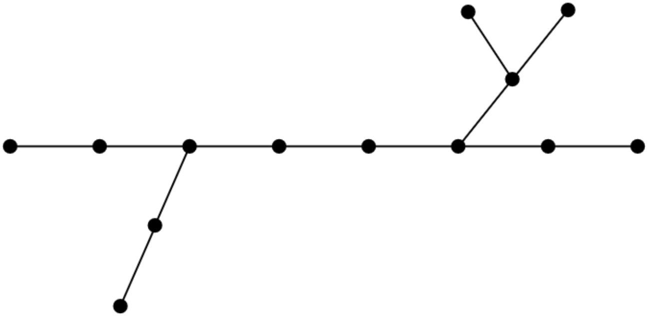
$$M = \textit{Tree metric}$$

$$T \; : \; V(T) = M \, , \;\; E(T) = \{n, n+1\}_n$$

ℓ_1 isometric Transportation cost space

It also works for any other weighted tree

Considering the shortest path distance



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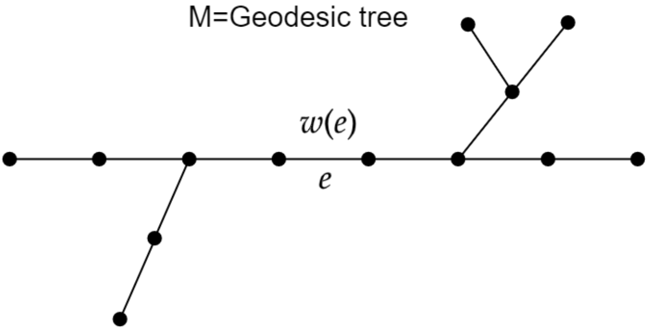

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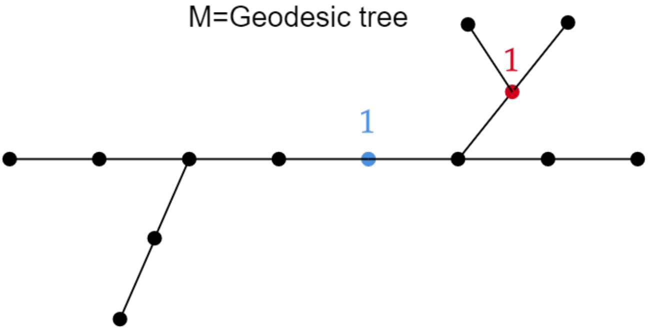

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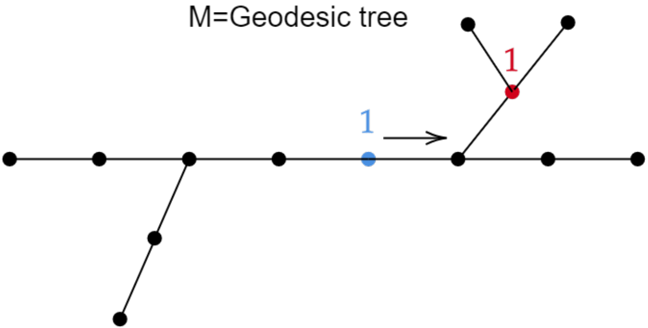

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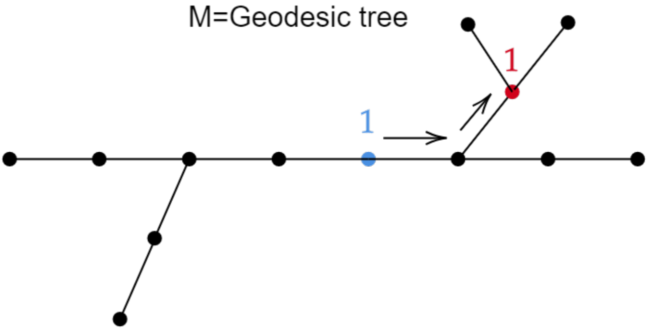
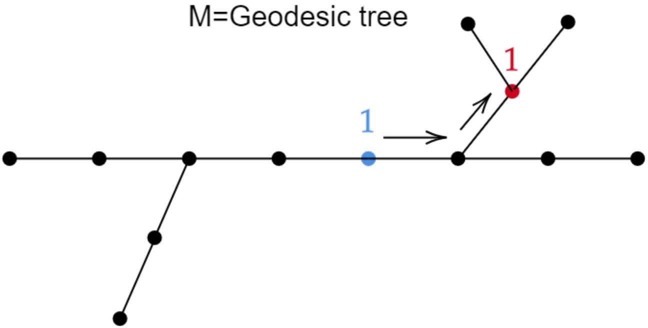


ℓ_1 isometric Transportation cost space

Theorem (Classic result - Godard?)

$\mathcal{F}(M)$ is isometric to ℓ_1^N if, and only if, M is a geodesic tree.



ℓ_1 isomorphic Transportation cost space

Question

What about isomorphically ℓ_1 Transportation cost spaces?

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Theorem (To appear)

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Let $(e_n)_{n=1}^N$ *be a basis for* $\mathcal{F}(M)$ *made of molecules, that is* $e_n \in span(\delta_x - \delta_y)$, *then TFAE:*

ℓ_1 isomorphic Transportation cost space

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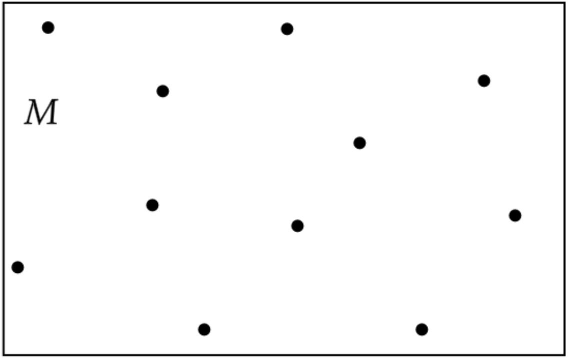
What if we consider more general bases? More general trees?

Tree metrics

Given a metric space M



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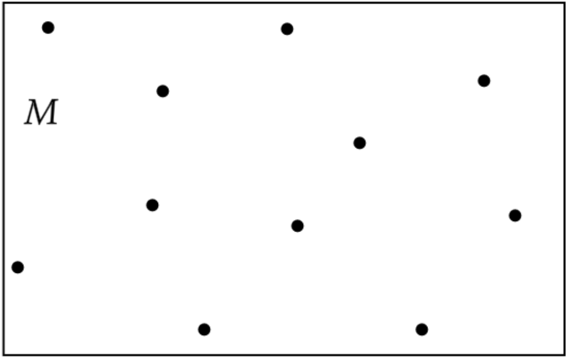

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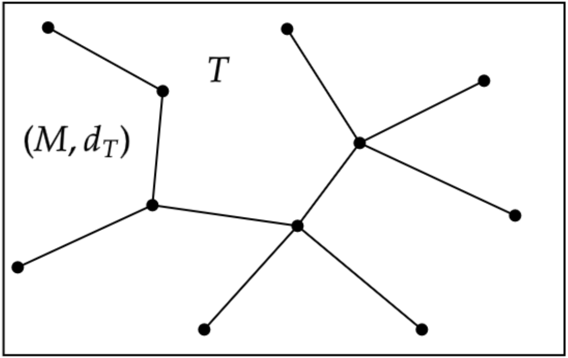
Given a metric space M

We consider the family of all possible trees $\mathcal{T}$

$d_T(x, y)$ = shortest path distance



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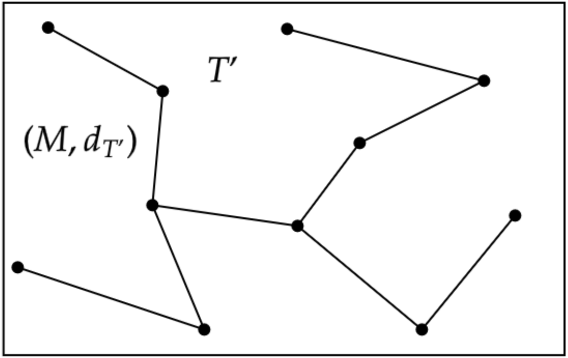

Stochastic tree metrics

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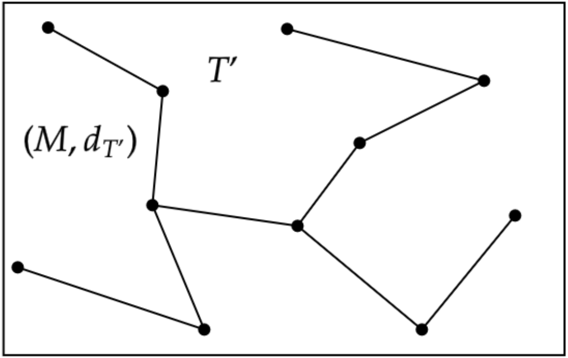
For $p \in \mathcal{P}(\mathcal{T})$, $\quad d_p(x,y) = \mathbb{E}_p(d_T(x,y)) = \sum_{T \in \mathcal{T}} p(T) d_T(x,y)$



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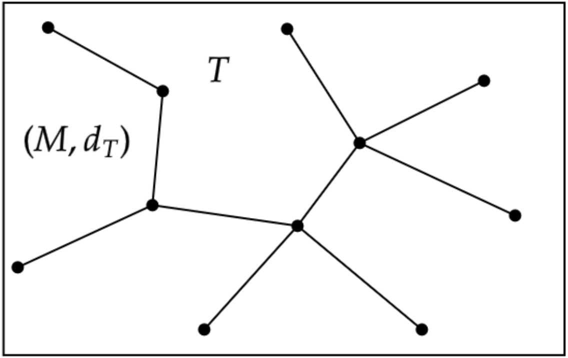

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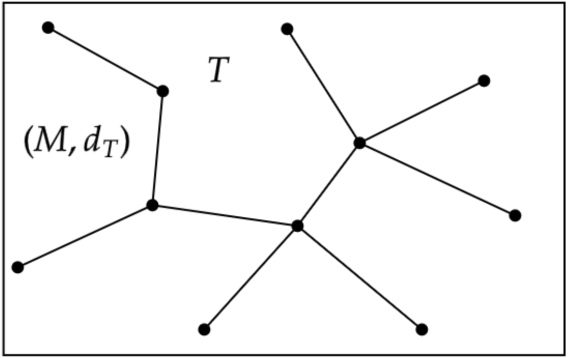
Theorem (Baudier, Motakis, Schlumprecht '22)

If M is C-isomorphic to a stochastic tree then $\mathcal{F}(M)$ is $8C$-complemented in ℓ_1.



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Question. Baudier and Schlumprecht '22

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Theorem (To appear)

Let $(e_n)_{n=1}^N$ be a stochastic basis of $\mathcal{F}(M)$, then for every $x, y \in M$ there is $p \in \mathcal{P}(\mathcal{T})$ such that

$$\mathbb{E}_p\big(d_T(x,y)\big) = \sum_{n=1}^{N} \big|e_n^*(\delta_x - \delta_y)\big| \|e_n\|.$$

Thank you for your attention!