The Banach-Saks rank of a separable weakly compact set

Víctor Olmos Prieto

New perspectives in Banach spaces and Banach lattices

Consider a family $\mathcal{A}$ of Banach spaces, or of subsets of Banach spaces. A rank or index on $\mathcal{A}$ is a map $\varrho : \mathcal{A} \to \mathbf{On}$. Usually they determine, knowing that a space forbids certain infinite structure, the moment when a natural process of building a copy of such a structure inside the space breaks.

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Derivation process
$B_X \to B'_X \to \ldots \to B_X^{(\alpha)}$

X

$\varrho(X) =$ *"number of steps of the derivation"*

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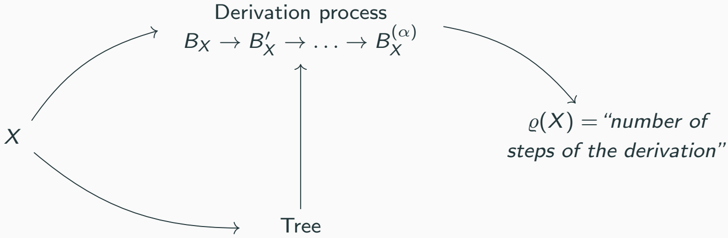

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and $P_\varepsilon^\alpha := \bigcap_{\beta<\alpha} P_\varepsilon^\beta$ for α limit. The Szlenk index of X is

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$$T(X, C) := \left\{(x_1, \ldots, x_n) \in X^{<\mathbb{N}} : (x_j)_{j=1}^n \text{ is } C\text{-equivalent to } (e_j)_{j=1}^n\right\}.$$

Bourgain's ℓ_1 rank of X is $\varrho_{\ell_1}(X) := \sup_{C\geq 1} o(T(X, C))$.

Mazur's and Banach-Saks' Theorem

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Theorem (Banach, Saks, 1930)

Every bounded sequence in $L^p(0,1)$, $1 < p < \infty$, has a Cesàro convergent subsequence.

Definition

A subset A of a Banach space X has the Banach-Saks property if every sequence in A has a Cesàro convergent subsequence. We say that X has the Banach-Saks property if its unit ball B_X has it.

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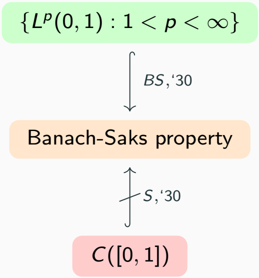

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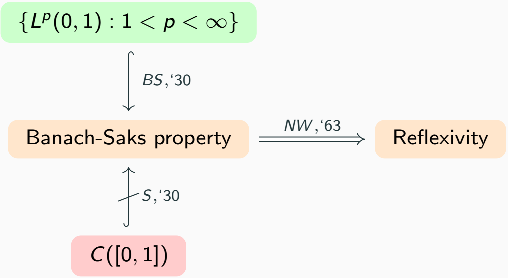

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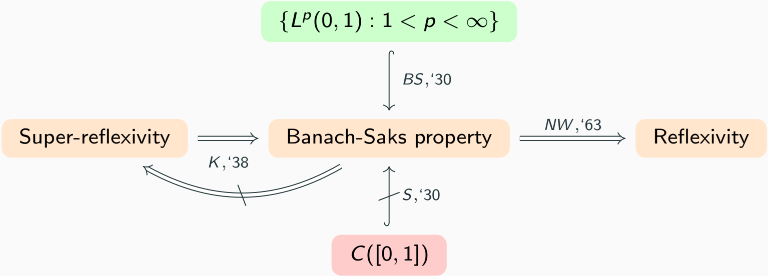

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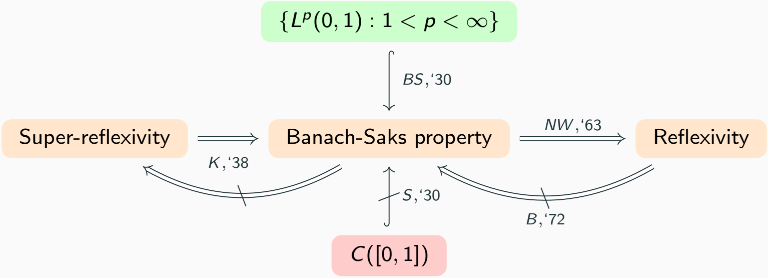

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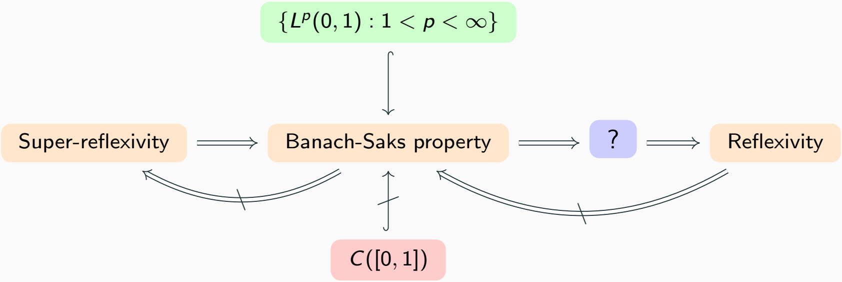

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- The coefficients a_n are *not important*.
- The supports s_k belong to the Schreier family

$$\mathcal{S} := \{s \subseteq \mathbb{N} : |s| \leq \min s\}.$$

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- One can construct a whole Schreier hierarchy $(\mathcal{S}_\xi)_{\xi < \omega_1}$ of ω^ξ-uniform families.

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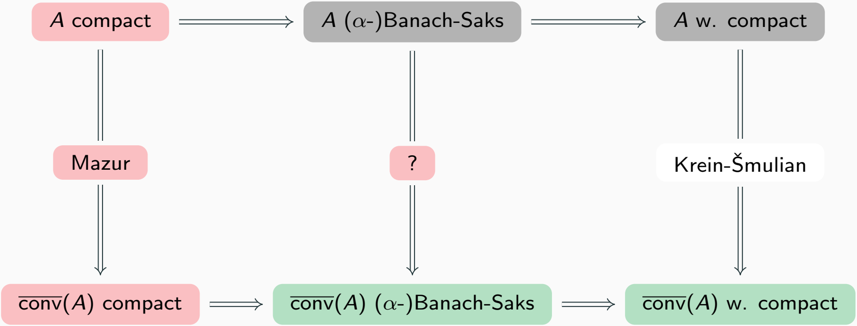
(ii) *$\varrho_{BS}(A)$ is always sum-indecomposable, i.e. of the form ω^α for some $\alpha < \omega_1$.*
- $\varrho_{BS}(A) \leq 1 \iff$ *A is relatively compact*
- $\varrho_{BS}(A) \leq \omega \iff$ *A is Banach-Saks*

The question



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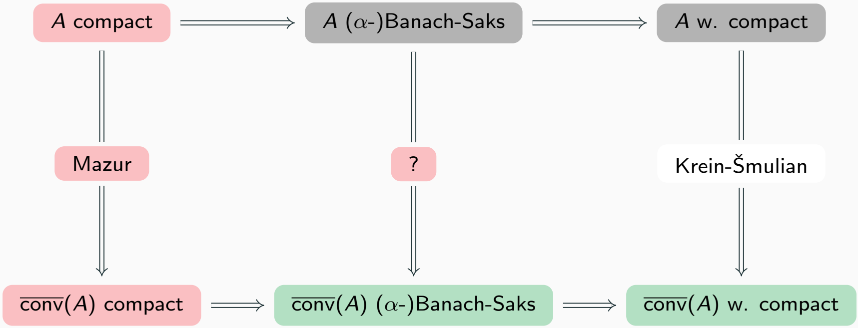

Theorem (López-Abad, Ruiz, Tradacete, 2013)

There exists a separable Banach-Saks set whose closed convex hull is not Banach-Saks.

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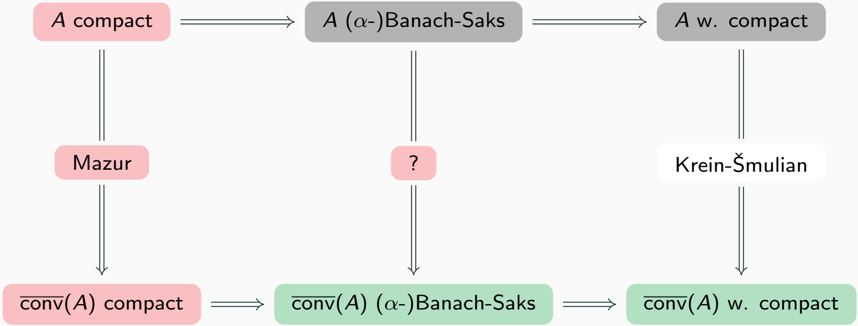

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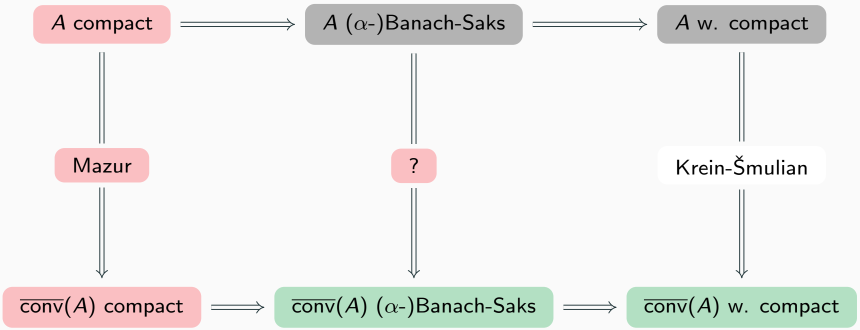

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Question 1

Is there an ordinal function $f : \omega_1 \to \omega_1$ such that $\varrho_{BS}(\overline{\mathrm{conv}}(A)) \leq f(\varrho_{BS}(A))$ for all separable weakly compact sets A?

A standard Borel space is a measurable space $(\Omega, \mathcal{A})$ whose σ-algebra is generated by a Polish topology. A subset $A \subseteq \Omega$ is analytic (or $\mathbf{\Sigma}^1_1$) if it is the image of another standard Borel space under a measurable map, and is co-analytic (or $\mathbf{\Pi}^1_1$) if $\Omega \setminus A$ is analytic.

Descriptive Set Theory and co-analytic ranks

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Let $(\Omega, \mathcal{A})$ be a standard Borel space and $A \in \mathbf{\Pi}_1^1(\Omega)$. A co-analytic rank on A is a map $\varrho : A \to \omega_1$ for which there exist two relations $\leq_{\mathbf{\Pi}}, \leq_{\mathbf{\Sigma}} \subseteq X \times X$ in $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_1^1$ respectively such that, for every $y \in A$ and every $x \in X$,

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Theorem (Boundedness Theorem for $\mathbf{\Pi}_1^1$-ranks)

Let $\varrho : A \to \omega_1$ be a co-analytic rank on a $\mathbf{\Pi}_1^1$ subset A of a standard Borel space.

(i) *For every $\xi < \omega_1$ the set $\{x \in A : \varrho(x) \leq \xi\}$ is Borel.*

(ii) *If $B \subseteq A$ is analytic then $\sup\{\varrho(x) : x \in B\} < \omega_1$.*

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Theorem (Boundedness Theorem for $\mathbf{\Pi}^1_1$-ranks)

Let $\varrho : A \to \omega_1$ be a co-analytic rank on a $\mathbf{\Pi}^1_1$ subset A of a standard Borel space.

(i) *For every $\xi < \omega_1$ the set $\{x \in A : \varrho(x) \leq \xi\}$ is Borel.*

(ii) *If $B \subseteq A$ is analytic then $\sup\{\varrho(x) : x \in B\} < \omega_1$.*

(iii) *If ϱ is another co-analytic rank on A, then there exists an increasing function $f : \omega_1 \to \omega_1$ such that $\varrho' \leq f(\varrho)$.*

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Is $\varrho_{BS} : RWC \to \omega_1$ a co-analytic rank?

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If the answer to Q2 is yes, then the answer to Q1 is also yes by the Boundedness Theorem: both ϱ_{BS} and $\varrho_{BS}(\overline{\text{conv}}(\cdot))$ would be co-analytic ranks, hence equivalent!

No!

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Idea of the proof.

- We define the uniform rank of a compact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ as

$$\mathrm{urk}(\mathcal{F}) := \sup\{\alpha < \omega_1 : \exists M \in [\mathbb{N}] \text{ s.t. } \mathcal{F} \cap \mathcal{P}(M) \text{ is } \alpha\text{-uniform on } M\}.$$

  ϱ_{BS} is related to the uniform rank of certain families. If ϱ_{BS} is co-analytic, so is urk.

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Again question 1

Is there an ordinal function $\phi : \omega_1 \to \omega_1$ such that $\varrho_{BS}(\overline{\text{conv}}(A)) \leq \phi(\varrho_{BS}(A))$ for all $A \in RWC$?

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Thank you for your attention!