Numerical index and the geometry of bounded linear operators

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New perspectives in Banach spaces and Banach lattices
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Classical numerical index

Numerical range (Bauer, Lumer, early 60's)

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- $v(\cdot)$ is a seminorm on $\mathcal{L}(X)$ and $v(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X)$

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- $n(X) = 1 \iff \max_{\theta\in\mathbb{T}} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$ for every $T \in \mathcal{L}(X)$

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- $n(X^*) \leqslant n(X)$
  and the inequality can be strict (Boyko–Kadets–Martín–Werner, 2007)

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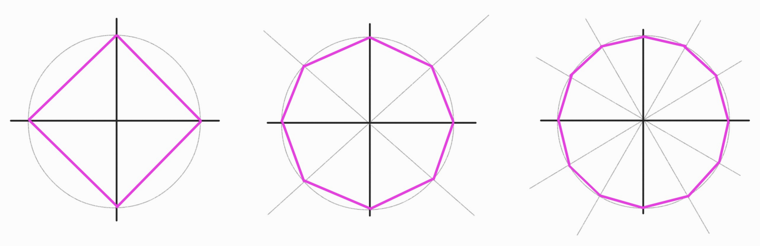
Open problem: Calculate $n(\ell_p)$ for $1 < p < \infty$, $p \neq 2$

Some examples in $\mathbb{R}^2$

- $X_n = (\mathbb{R}^2, \|\cdot\|_n)$ such that

$$\mathrm{ext}(B_{X_n}) = \left\{ \left(\cos\left(\frac{k\pi}{2n}\right),\ \sin\left(\frac{k\pi}{2n}\right) \right) : k = 1, 2, \ldots, 4n \right\}.$$

Then, $n(X_n) = \tan\left(\frac{\pi}{4n}\right)$ (Martín–Merí, 2007)



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$$\|(x,y)\|_\xi = \max\left\{ |x|,\ |y|,\ \frac{|x|+|y|}{1+\xi} \right\}$$

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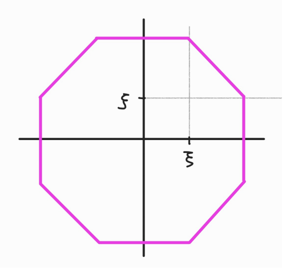

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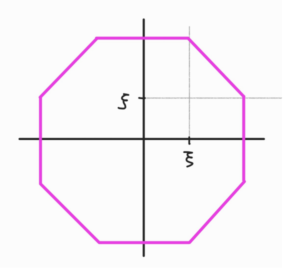

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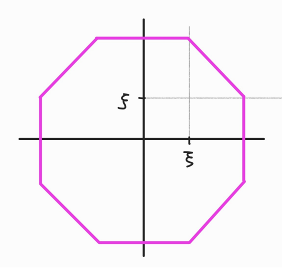
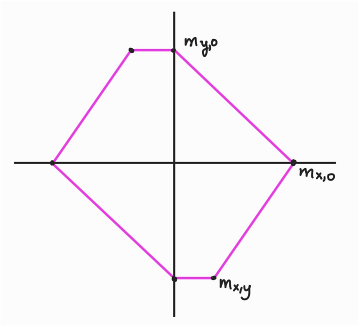


- $X = (\mathbb{R}^2, \|\cdot\|)$, $\|\cdot\|$ absolute and symmetric.

In "many cases", $n(X) = v\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (Merí–Q., 2021)

- $M = (\{x, y, 0\}, d)$ metric space with $d(x, y) \geqslant d(x, 0) \geqslant d(y, 0)$. Then,
  - $n(\mathcal{F}(M)) = 1$ if M is aligned,
  - $n(\mathcal{F}(M)) = \max\left\{ \dfrac{d(x,0)G_y(x,0)}{d(x,y)G_0(x,y) + d(y,0)G_x(y,0)}, \dfrac{d(x,0)}{d(x,y), d(y,z)} \right\}$ otherwise.



(Cobollo–Guirao–Montesinos, 2024)

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Question (real case): Let $1 < p < \infty$, $p \neq 2$

$$n(\ell_p^2) = M_p = \max_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ?$$

Theorem (Merí–Q., 2021)

Let $p \in \left[\frac{3}{2}, 3\right]$. Then

$$n(\ell_p^2) = M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

★ **[Monika–Zheng, 2022]:** $n(\ell_p^2) = M_p$ for $1.4547 \lesssim p \lesssim 3.1993$.

Theorem (Merí–Q., 2024)

Let $p \in \left[\frac{6}{5}, \frac{3}{2}\right] \cup [3, 6]$. Then

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If X has a long biorthogonal system, then

$$\{n(X,|\cdot|)\colon |\cdot| \text{ equivalent norm on } X\} \supseteq \begin{cases} [0,1) & \text{real case} \\ [1/\mathrm{e},1) & \text{complex case} \end{cases}$$

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- If X^* is smooth or strictly convex, then $X = \mathbb{K}$.
  (Kadets–Martín–Merí–Payá, 2009)
- If X is smooth and B_X has strongly exposed points, then $X = \mathbb{K}$.
  (Martín–Merí–Q.–Roy–Sain, 2024)

Numerical index with respect to an operator

(Bonsall–Duncan, 1971)

X Banach space. Then, for every $T \in \mathcal{L}(X)$,

$$\overline{\mathrm{conv}}(V(T)) = \{\phi(T)\colon \phi \in \mathcal{L}(X)^*,\ \|\phi\| = \phi(\mathrm{Id}) = 1\}$$

Consequently, $v(T) = \max\{|\phi(T)|\colon \phi \in \mathcal{L}(X)^*,\ \|\phi\| = \phi(\mathrm{Id}) = 1\}$

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★ $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$, how do we define $\widetilde{V}_G(T)$?

Extending the concept of numerical range

(Bonsall–Duncan, 1971)

X Banach space. Then, for every $T \in \mathcal{L}(X)$,

$$\overline{\text{conv}}(V(T)) = \{\phi(T) \colon \phi \in \mathcal{L}(X)^*, \ \|\phi\| = \phi(\text{Id}) = 1\}$$

Consequently, $v(T) = \max\{|\phi(T)| \colon \phi \in \mathcal{L}(X)^*, \ \|\phi\| = \phi(\text{Id}) = 1\}$

Intrinsic numerical range

X Banach space, $T \in \mathcal{L}(X)$

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Intrinsic numerical range with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

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Extending the concept of numerical range

X Banach space, $T \in \mathcal{L}(X)$,

$$V(T) = \{x^*(Tx) \colon x \in S_X,\, x^* \in S_{X^*},\, x^*(\mathrm{Id}\, x) = 1\}$$

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Approximated spatial numerical range with respect to G (Ardalani, 2014)

X, Y Banach spaces, $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X, Y)$

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Both concepts produce the same numerical radius!

Numerical radius with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

$$v_G(T) := \sup\{|\lambda|\colon \lambda \in V_G(T)\} = \sup\{|\lambda|\colon \lambda \in \widetilde{V}_G(T)\}.$$

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- $v_G(\cdot)$ is a seminorm on $\mathcal{L}(X,Y)$ and $v_G(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X,Y)$.
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Numerical index with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$

$$n_G(X,Y) := \inf\{v_G(T) \colon T \in S_{\mathcal{L}(X,Y)}\}.$$

Equivalently, $n_G(X,Y) = \max\{k \geqslant 0 \colon k\|T\| \leqslant v_G(T)\ \forall\, T \in \mathcal{L}(X,Y)\}$.

Numerical index with respect to an operator

Numerical radius with respect to G

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We recover the classical concepts (Ardalani, 2014)

$$V_{\mathrm{Id}}(T) = \overline{V(T)}, \qquad v_{\mathrm{Id}}(T) = v(T), \qquad n_{\mathrm{Id}}(X,X) = n(X).$$

Proposition

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $\lambda \in (0,1]$. TFAE:

1. $n_G(X,Y) \geqslant \lambda$.
2. $\max_{\theta\in\mathbb{T}} \|G + \theta T\| \geqslant 1 + \lambda\|T\|$ for every $T \in \mathcal{L}(X,Y)$.

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Spear operator (Ardalani, 2014)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$.

$$G \textit{ spear operator } \overset{\text{def}}{\Longleftrightarrow} \max_{\theta\in\mathbb{T}} \|G + \theta T\| = 1 + \|T\| \text{ for every } T \in \mathcal{L}(X,Y)$$
$$\Longleftrightarrow v_G(T) = \|T\| \text{ for every } T \in \mathcal{L}(X,Y).$$

Examples of spear operators (Kadets–Martín–Merí–Pérez, 2018)

- The identity operator on $C(K)$, $L_1(\mu)$...
- Inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.
- Fourier transform $\mathcal{F}\colon L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$.

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Isomorphic and isometric consequences (Kadets–Martín–Merí–Pérez, 2018)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ spear operator.

- If X is real and $\dim G(X) = \infty$, then $X^* \supset \ell_1$.
- If X^* is strictly convex or smooth, then $X = \mathbb{K}$.
- If Y^* is strictly convex, then $Y = \mathbb{K}$.
- If B_X has strongly exposed points and Y is smooth, then $Y = \mathbb{K}$

(Martín–Merí–Q.–Roy–Sain, 2024)

X, Y Banach spaces

$$\mathcal{N}(\mathcal{L}(X,Y)) := \{n_G(X,Y) \colon G \in \mathcal{L}(X,Y),\ \|G\| = 1\}$$

Numerical index with respect to an operator. Set of values

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Theorem (Kadets–Martín–Merí–Pérez–Q., 2020)

X or Y Hilbert with $\dim \geqslant 2$:

- Real case: $\mathcal{N}(\mathcal{L}(X,Y)) = \{0\}$. In particular, $\mathcal{N}(\mathcal{L}(H)) = \{0\}$.
- Complex case: $\mathcal{N}(\mathcal{L}(X,Y)) \subseteq [0, 1/2]$.

H_1, H_2 complex Hilbert spaces with $\dim \geqslant 2$:

- $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0, 1/2\}$ if H_1 and H_2 are isometrically isomorphic.
- $\mathcal{N}(\mathcal{L}(H_1,H_2)) = \{0\}$ otherwise.

Proposition (Kadets–Martín–Merí–Pérez–Q., 2020)

For $1 < p < \infty$, X, Y Banach spaces, $M_p = \sup_{t\in[0,1]} \frac{|t^{p-1}-t|}{1+t^p}$,

$$\mathcal{N}(\mathcal{L}(X,\ell_p)) \subseteq [0, M_p] \qquad \text{and} \qquad \mathcal{N}(\mathcal{L}(\ell_p,Y)) \subseteq [0, M_p] \qquad \text{(real case)}$$

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(Duncan–McGregor–Pryce–White, 1970)

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Question:

$$\{n_G(X,X)\colon X \text{ (real or complex) Banach space}, G \in \mathcal{L}(X), \|G\| = 1\} = [0,1]?$$

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Theorem (Kadets–Martín–Merí–Pérez–Q., 2020)

There exist (*real and complex*) Banach spaces X such that $\mathcal{N}(\mathcal{L}(X)) = [0, 1]$.