Metric characterization of Dunford-Pettis Lipschitz operators

Joint work with G. Flores, M. Jung, G. Lancien, C. Petitjean and A. Procházka

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Setting

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Theorem

Let $f\colon M \to N$ be a Lipschitz function. Then there exists a linear operator $\widehat{f}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ {\scriptstyle \delta_M}\downarrow & & \downarrow{\scriptstyle \delta_N} \\ \mathcal{F}(M) & \xrightarrow[\widehat{f}]{} & \mathcal{F}(N). \end{array}$$

Moreover, we have that $\|\widehat{f}\| = \mathrm{Lip}(f)$.

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	weakly compact operator	Abbar, Coine, Petitjean (2023, Proc. Roy. Soc. Edim.)

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curve-flat	Completely continuous (or Dunford-Pettis) operator	FJLPPQ

Main result

Theorem (FJLPPQ)

Let M be a compact metric space, and let $f\colon M \to N$ be a curve-flat Lipschitz function (i.e. the derivative of f is 0 a.e. along curves in M).

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Lemma (Bate 2020, Acta Math.)

If $g\colon M \to \mathbb{R}$ is curve-flat, for every $\varepsilon > 0$ there exists an ε-locally flat function $h\colon M \to \mathbb{R}$ such that $\|h - g\|_\infty < \varepsilon$ and $\mathrm{Lip}(h) = \mathrm{Lip}(g)$.

General case: not just compact

Theorem (FJLPPQ)

Let M be a complete metric space. Then $f : M \to N$ is curve-flat if and only if $\widehat{f} \colon \mathcal{F}(M) \to \mathcal{F}(N)$ is completely continuous.

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Theorem (Aliaga, Petitjean, Procházka (2021, Studia Math.)

If $W \subset \mathcal{F}(M)$ is weakly precompact, for every $\varepsilon > 0$ there exist a compact $K \subset M$ and a linear map $T \mathrm{span}(W) \to \mathcal{F}(K)$ such that

- $\|\mu - T\mu\| \leq \varepsilon$ for all $\mu \in W$, and
- there is a sequence of bounded linear operators $T_k\colon \mathcal{F}(M) \to \mathcal{F}(M)$ such that $T_k \to T$ uniformly on W.

Thank you for you attention!