Generic nonexpansive Hilbert space mappings

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- …a closed, convex set $C \subseteq H$,
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- …the topology of pointwise convergence on $\mathcal{N}(C)$.

We are interested in the fixed point of the generic $f \in \mathcal{N}(C)$.

A few questions

1. Why pointwise convergence?
2. Why separability?
3. Why Hilbert spaces?

Why pointwise convergence?

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Most natural topologies on $\mathcal{N}(C)$ have already been thoroughly investigated. These include:

- the topology of uniform convergence, if C is bounded;
- the topology of uniform convergence on bounded sets, if C is unbounded.

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But strict contractions are very few!

1989 de Blasi and Myjak show it for Hilbert spaces.

2015 Bargetz and Dymond show it for Banach spaces.

2017 Bargetz, Dymond and Reich show it for a larger class of geodesic metric spaces.

Uniform convergence on bounded sets

X complete, unbounded hyperbolic metric space.

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But Rakotch contractions are very few!

2012 Strobin shows it for Hilbert spaces.

2023 Bargetz, Reich and Thimm show it for complete, hyperbolic metric spaces.

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If (X, ρ) is a complete metric space and $\{\theta_n\}_{n=1}^{\infty}$ is a dense subset, then the distance $d \colon \mathcal{N}(X) \times \mathcal{N}(X) \to \mathbb{R}$ given by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho\big(f(\theta_n), g(\theta_n)\big)}{1 + \rho\big(f(\theta_n), g(\theta_n)\big)}$$

is complete and metrises the topology of pointwise convergence.

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Two nice theorems make our job easier in Hilbert spaces.

Theorem (Browder–Göhde–Kirk)

If C is a closed, convex, bounded set in a uniformly convex Banach space, every nonexpansive mapping on C has a fixed point.

Why Hilbert spaces?

Honest answer: math is hard. . .

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Theorem (Kirszbraun–Valentine)

Let H_1, H_2 be Hilbert spaces. Let $A \subseteq H_1$ be any subset and let $f\colon A \to H_2$ be a Lipschitz mapping. Then f admits a Lipschitz extension $F\colon H_1 \to H_2$ with the same Lipschitz constant.

The takeaway

For the first time we observe two opposite behaviours depending on what set C is given.

Meet somewhat bounded set

Definition

A convex set C in an infinite-dimensional Hilbert space H is somewhat bounded if there are $z_0 \in C$, a finite-dimensional subspace $F \subset H$ and $\alpha > 0$ such that $\alpha B_F \subset C - z_0$ and $F^\perp \cap (C - z_0)$ is bounded.

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A convex set C in an infinite-dimensional Hilbert space H is *somewhat bounded* *if there are $z_0 \in C$, a finite-dimensional subspace $F \subset H$ and $\alpha > 0$ such that $\alpha B_F \subset C - z_0$ and $F^\perp \cap (C - z_0)$ is bounded.*

C is *totally unbounded* *if C is not somewhat bounded.*

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C is totally unbounded if C is not somewhat bounded.

Examples:

- Infinite-dimensional, closed subspaces of H are totally unbounded.
- $F + B_H$ is somewhat bounded if F is finite-dimensional.

The two main theorems

Theorem (R., Thimm, 2024+)

Let C be a closed, convex, somewhat bounded set in a separable, infinite-dimensional Hilbert space H. Then there is a dense open set $\mathcal{R} \subset \mathcal{N}(C)$ such that every $f \in \mathcal{R}$ has a fixed point.

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Theorem (R., Thimm, 2024+)

Let C be a totally unbounded set in a separable, infinite-dimensional, Hilbert space H. Then there is a dense, G_δ set $\mathcal{R}$ such that every $f \in \mathcal{R}$ does not have any fixed points.

A topological 0–1 law

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Corollary

Let C be a closed, convex set in a separable, infinite-dimensional Hilbert space H and let $\mathcal{F} \subset \mathcal{N}(C)$ be the set of all nonexpansive mappings on C with a fixed point. Then either $\mathcal{F}$ is meagre or it contains a dense open set.

Fixed points are on the boundary

Theorem (R., Thimm, 2024+)

Let C be a closed, convex, somewhat bounded set in a separable, infinite-dimensional Hilbert space H. Then there is a dense G_δ set $\mathcal{R} \subset \mathcal{N}(C)$ such that, for every $f \in \mathcal{R}$, there are no fixed points of f in $\operatorname{int} C$.

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Corollary

Let C be a closed, strictly convex, somewhat bounded set in a separable, infinite-dimensional Hilbert space H. Then there is a dense G_δ set $\mathcal{R} \subset \mathcal{N}(C)$ such that every $f \in \mathcal{R}$ has a unique fixed point.

Convergence of the iterates

Theorem (R., Thimm, 2024+)

Let C be a closed, somewhat bounded LUR set in a separable, infinite-dimensional Hilbert space. Then there exists a dense G_δ set $\mathcal{R} \subset \mathcal{N}(C)$ such that every $f \in \mathcal{R}$ has a unique fixed point and the sequence of iterates $(f^k(x))_{k\in\mathbb{N}}$ converges to the fixed point of f for every $x \in C$.

The key lemma

Lemma

Let C be a closed, convex, somewhat bounded set in a separable, infinite-dimensional Hilbert space H. Then there is a dense G_δ set $\mathcal{R} \subset \mathcal{N}(C)$ such that for every $f \in \mathcal{R}$ and every $x \in C$ we have $\operatorname{dist}(f^k(x), \partial C) \to 0$ as $k \to \infty$.

Open questions

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Wait until the open problem session :)

Gracias!