Algebraic structures of non-norm-attaining operators

Daniel L. Rodríguez-Vidanes
Department of Applied Mathematis to Industrial Engineering
School of Engineering and Industrial Design
Technical University of Madrid
Research Group: Mathematical Analysis and Applications

Joint work with S. Dantas, J. Falcó and M. Jung

Summer School: New Perspectives in Banach spaces and Banach Lattices

July 8, 2024

escuela técnica superior de ingeniería y diseño industrial

UNIVERSIDAD POLITÉCNICA DE MADRID

Outline

1. Lineability and norm-attaining
2. New results

Lineability and norm-attaining

Classical lineability

Definition

Let V be a topological vector space defined over $\mathbb{R}$ or $\mathbb{C}$, $A \subset V$ and κ a cardinal number. We say that A is:

- κ-*lineable* if $A \cup \{0\}$ contains a subspace of V of dimension κ.
- κ-*spaceable* if $A \cup \{0\}$ contains a closed subspace of V of dimension κ.

Modern lineability

Definition

Let V be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, $A \subset V$ and $\alpha \leq \beta$ two cardinal numbers. We say that A is (α, β)-spaceable if A is α-lineable and for every α-dimensional vector subspace V_α of V contained in $A \cup \{0\}$, there is a closed β-dimensional vector subspace V_β of V such that

$$V_\alpha \subseteq V_\beta \subseteq A \cup \{0\}.$$

Modern lineability

Definition

Let V be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, $A \subset V$ and $\alpha \leq \beta$ two cardinal numbers. We say that A is (α, β)-spaceable if A is α-lineable and for every α-dimensional vector subspace V_α of V contained in $A \cup \{0\}$, there is a closed β-dimensional vector subspace V_β of V such that

$$V_\alpha \subseteq V_\beta \subseteq A \cup \{0\}.$$

Remark

If A is (α, α)-spaceable, then A is α-spaceable, but **the converse is not true in general**. *For instance, $L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]$ is $\mathfrak{c}$-spaceable, but not $(\mathfrak{c}, \mathfrak{c})$-spaceable, where $\mathfrak{c}$ denotes the cardinality of the continuum.*

- *Botelho, Fávaro, Pellegrino and Seoane-Sepúlveda (2012).*
- *Fávaro, Pellegrino, Raposo Jr. and Ribeiro (2024).*

Norm-attaining operators

Definition

Let X and Y be Banach spaces. A continuous linear operator $T\colon X \to Y$ is norm-attaining if there exists $x \in S_X$ such that $\|T\| = \|Tx\|$. We denote the set of norm-attaining operators from X to Y by $\mathrm{NA}(X, Y)$. If $Y = \mathbb{K}$, we simply denote $\mathrm{NA}(X)$.

Norm-attaining functionals vs. lineability

Proposition

$$\mathrm{NA}(c_0) = c_{00} \leq \ell_1.$$

Norm-attaining functionals vs. lineability

Proposition

$$\mathrm{NA}(c_0) = c_{00} \leq \ell_1.$$

Proposition

$$\mathrm{NA}(\ell_1) = \left\{ x \in \ell_\infty : \|x\|_\infty = \max_{n \in \mathbb{N}} |x_n| \right\}$$

contains c_0 but is not a subspace of ℓ_∞.

An extremely non-lineable norm-attaining set

Question by Bandyopadhyay and Godefroy (2006)

Let X be an infinite dimensional Banach space. Is $\mathrm{NA}(X)$ 2-lineable?

An extremely non-lineable norm-attaining set

Question by Bandyopadhyay and Godefroy (2006)

Let X be an infinite dimensional Banach space. Is $\mathrm{NA}(X)$ 2-lineable?

Theorem (Rmoutil (2017))

There is a Banach space X such that $\mathrm{NA}(X)$ is not 2-lineable.

An extremely non-lineable norm-attaining set

Question by Bandyopadhyay and Godefroy (2006)

Let X be an infinite dimensional Banach space. Is $\mathrm{NA}(X)$ 2-lineable?

Theorem (Rmoutil (2017))

There is a Banach space X such that $\mathrm{NA}(X)$ is not 2-lineable.

Remark

Rmoutil's counterexample is Read's space.

On renormings

Theorem (García-Pacheco, Puglisi (2018))

Every real infinite-dimensional Banach space X admits a renorming such that $\mathrm{NA}(X, \|\cdot\|)$ is $\aleph_0$-lineable, where $\aleph_0$ denotes the cardinality of $\mathbb{N}$.

General results on spaceability

Theorem (Bandyopadhyay, Godefroy (2006))

Let X be a real Banach space such that X is a weakly compact generated Asplund space or X^* is separable. Then TFAE:

(i) *There is a renorming of X such that $\mathrm{NA}(X, \|\cdot\|)$ is $\mathfrak{c}$-spaceable.*

(ii) *X^* contains an infinite-dimensional reflexive subspace.*

General results on spaceability

Theorem (Bandyopadhyay, Godefroy (2006))

Let X be a real Banach space such that X is a weakly compact generated Asplund space or X^* is separable. Then TFAE:

(i) *There is a renorming of X such that $\mathrm{NA}(X, \|\cdot\|)$ is $\mathfrak{c}$-spaceable.*

(ii) *X^* contains an infinite-dimensional reflexive subspace.*

Open question by Bandyopadhyay and Godefroy (2006)

Is the above theorem true for any Asplund space?

General results on spaceability and Banach lattices

Theorem (Cheng, Luo (2018))

Let X be a real Asplund space such that

(a) *X is a Banach lattice or*

(b) *X is the quotient of $\mathcal{C}(K)$ for some compact Hausdorff space K.*

Then TFAE:

(i) *There is a renorming of X such that $\mathrm{NA}(X, \|\cdot\|)$ is $\mathfrak{c}$-spaceable.*

(ii) *X^* contains an infinite-dimensional reflexive subspace.*

Non-norm-attaining functionals vs. lineability

Theorem (Acosta, Aizpuru, Aron, García (2007))

If K is an infinite compact Hausdorff topological space and $\mathcal{C}(K)$ takes values in $\mathbb{R}$, then $\mathcal{C}(K)^* \setminus \mathrm{NA}(\mathcal{C}(K))$ is $\aleph_0$-lineable. Moreover, if K has a non-trivial convergent sequence, then $\mathcal{C}(K)^* \setminus \mathrm{NA}(\mathcal{C}(K))$ is $\mathfrak{c}$-spaceable.

Norm-attaining operators vs. lineability

Proposition

If $X \neq \{0\}$ and Y are Banach spaces, then Y is isometrically embedded in $\mathrm{NA}(X, Y)$.

Norm-attaining operators vs. lineability

Proposition

If $X \neq \{0\}$ and Y are Banach spaces, then Y is isometrically embedded in $\mathrm{NA}(X, Y)$.

Proof.

By Hahn-Banach theorem, there exists $x^* \in \mathrm{NA}(X, \mathbb{K})$ with $\|x^*\| = 1$. The map $y \mapsto x^* \otimes y$ is as needed. □

New results

Lindenstrauss' counterexample

Theorem (Bishop, Phelps (1961))

The set NA(X) *is dense in* X^* *for any Banach space* X.

Lindenstrauss' counterexample

Theorem (Bishop, Phelps (1961))

The set $\mathrm{NA}(X)$ *is dense in* X^* *for any Banach space* X.

Theorem (Lindenstrauss (1963))

There exist Banach spaces X *and* Y *such that* $\mathcal{L}(X, Y) \setminus \overline{\mathrm{NA}(X, Y)}$ *is non-empty.*

Lineability on Lindenstrauss' counterexample

Theorem (Dantas, Falcó, Jung, R-V (2023))

Let Γ *be an infinite set and* Y *an strictly convex renorming of* $c_0(\Gamma)$*. Then*

$$\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)}$$

is $2^{|\Gamma|}$*-spaceable in* $\mathcal{L}(c_0(\Gamma), Y)$*.*

Fichtenholz-Kantorovich-Hausdorff Theorem

Definition (Family of indepedendent subsets)

Let Γ be an infinite set. We say that $\Omega \subset \mathcal{P}(\Gamma)$ is a family of independent subsets of Γ if for any finite sequences $A_1, \ldots, A_n, B_1, \ldots, B_m \in \Omega$ pairwise distinct it yields that

$$|A_1 \cap \cdots \cap A_n \cap (\Gamma \setminus B_1) \cap \cdots \cap (\Gamma \setminus B_m)| = |\Gamma|.$$

Fichtenholz-Kantorovich-Hausdorff Theorem

Definition (Family of indepedendent subsets)

Let Γ be an infinite set. We say that $\Omega \subset \mathcal{P}(\Gamma)$ is a family of independent subsets of Γ if for any finite sequences $A_1, \ldots, A_n, B_1, \ldots, B_m \in \Omega$ pairwise distinct it yields that

$$|A_1 \cap \cdots \cap A_n \cap (\Gamma \setminus B_1) \cap \cdots \cap (\Gamma \setminus B_m)| = |\Gamma|.$$

Theorem (Fichtenholz-Kantorovich-Hausdorff theorem)

Let Γ be an infinite set. There is a family Ω of independent subsets of Γ with cardinality $2^{|\Gamma|}$.

Fichtenholz-Kantorovich-Hausdorff Theorem

Definition (Family of indepedendent subsets)

Let Γ be an infinite set. We say that $\Omega \subset \mathcal{P}(\Gamma)$ is a family of independent subsets of Γ if for any finite sequences $A_1, \ldots, A_n, B_1, \ldots, B_m \in \Omega$ pairwise distinct it yields that

$$|A_1 \cap \cdots \cap A_n \cap (\Gamma \setminus B_1) \cap \cdots \cap (\Gamma \setminus B_m)| = |\Gamma|.$$

Theorem (Fichtenholz-Kantorovich-Hausdorff theorem)

Let Γ be an infinite set. There is a family Ω of independent subsets of Γ with cardinality $2^{|\Gamma|}$.

We will call Fichtenholz-Kantorovich-Hausdorff theorem by FKH.

Idea of the proof

For any $\gamma \in \Gamma$, denote $e_\gamma \in c_0(\Gamma)$ as:

$$e_\gamma(\xi) := \begin{cases} 1 & \text{if } \xi = \gamma, \\ 0 & \text{otherwise} \end{cases}$$

for each $\xi \in \Gamma$.

Idea of the proof

For any $\gamma \in \Gamma$, denote $e_\gamma \in c_0(\Gamma)$ as:

$$e_\gamma(\xi) := \begin{cases} 1 & \text{if } \xi = \gamma, \\ 0 & \text{otherwise} \end{cases}$$

for each $\xi \in \Gamma$.

By FKH, there exists Ω a family of independent subsets of Γ with cardinality $2^{|\Gamma|}$.

Idea of the proof

For every $F \in \Omega$, define $T_F\colon c_0(\Gamma) \to Y$ as follows: for any $x = \sum_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$, let

$$T_F(x) = \sum_{\gamma\in\Gamma} x_\gamma T_F(e_\gamma),$$

with

$$T_F(e_\gamma) = \begin{cases} e_\gamma & \text{if } \gamma \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Idea of the proof

For every $F \in \Omega$, define $T_F \colon c_0(\Gamma) \to Y$ as follows: for any $x = \sum_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$, let

$$T_F(x) = \sum_{\gamma\in\Gamma} x_\gamma T_F(e_\gamma),$$

with

$$T_F(e_\gamma) = \begin{cases} e_\gamma & \text{if } \gamma \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Then the closed subspace

$$\overline{\operatorname{span}}\{T_F \colon F \in \Omega\} \subset \left(\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)}\right) \cup \{0\}$$

and has dimension $2^{|\Gamma|}$.

Idea of the proof

Step 1: T_F is well-defined and bounded for any $F \in \Omega$.

Idea of the proof

Step 1: T_F is well-defined and bounded for any $F \in \Omega$.

Step 2: $\{T_F : F \in \Omega\}$ is linearly independent by FKH.

Idea of the proof

Step 1: T_F is well-defined and bounded for any $F \in \Omega$.

Step 2: $\{T_F : F \in \Omega\}$ is linearly independent by FKH.

Step 3: The non-zero finite linear combinations of $\{T_F : F \in \Omega\}$ are in $\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\text{NA}(c_0(\Gamma), Y)}$ by FKH and the fact that each $S \in \text{NA}(c_0(\Gamma), Y)$ depends only on finitely many e_γ's.

Idea of the proof

Step 1: T_F is well-defined and bounded for any $F \in \Omega$.

Step 2: $\{T_F : F \in \Omega\}$ is linearly independent by FKH.

Step 3: The non-zero finite linear combinations of $\{T_F : F \in \Omega\}$ are in $\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)}$ by FKH and the fact that each $S \in \mathrm{NA}(c_0(\Gamma), Y)$ depends only on finitely many e_γ's.

Step 4: Each $S \in \overline{\mathrm{span}}\{T_F : F \in \Omega\}$ cannot be approximated by operators in $\mathrm{NA}(c_0(\Gamma), Y)$ by FKH.

Idea of the proof

Step 1: T_F is well-defined and bounded for any $F \in \Omega$.

Step 2: $\{T_F : F \in \Omega\}$ is linearly independent by FKH.

Step 3: The non-zero finite linear combinations of $\{T_F : F \in \Omega\}$ are in $\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)}$ by FKH and the fact that each $S \in \mathrm{NA}(c_0(\Gamma), Y)$ depends only on finitely many e_γ's.

Step 4: Each $S \in \overline{\mathrm{span}}\{T_F : F \in \Omega\}$ cannot be approximated by operators in $\mathrm{NA}(c_0(\Gamma), Y)$ by FKH.

Step 5: $\dim\left(\overline{\mathrm{span}}\{T_F : F \in \Omega\}\right) = 2^{|\Gamma|}$.

Modern lineability on Lindenstrauss' counterexample

Corollary (Dantas, Falcó, Jung, R-V (2023))

Let Γ *be an infinite set,* Y *an strictly convex renorming of* $c_0(\Gamma)$ *and* $\aleph_0 \leq \alpha \leq 2^{|\Gamma|}$ *a cardinal number. Then*

$$\mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}(c_0(\Gamma), Y)}$$

is not (α, β)*-spaceable in* $\mathcal{L}(c_0(\Gamma), Y)$ *regardless of the cardinal number* $\beta \geq \alpha$.

Idea of the proof

Theorem (Fávaro, Pellegrino, Raposo Jr., Ribeiro (2024))

Let $\alpha \geq \aleph_0$ and V be an F-space. Let A, B be subsets of V such that A is α-lineable and B is 1-lineable. If $A \cap B = \varnothing$ and $A + B \subset A$, then A is not (α, β)-spaceable, regardless of the cardinal number β.

Idea of the proof

Theorem (Fávaro, Pellegrino, Raposo Jr., Ribeiro (2024))

Let $\alpha \geq \aleph_0$ and V be an F-space. Let A, B be subsets of V such that A is α-lineable and B is 1-lineable. If $A \cap B = \varnothing$ and $A + B \subset A$, then A is not (α, β)-spaceable, regardless of the cardinal number β.

For

$$A := \mathcal{L}(c_0(\Gamma), Y) \setminus \overline{\mathrm{NA}_G(c_0(\Gamma), Y)}$$

and

$$B := \overline{\mathrm{NA}(c_0(\Gamma), Y)},$$

we have that

$$A + B \subset A.$$

Lindenstrauss property B

Definition

A Banach space Y satisfies *Lindenstrauss property B* if $\mathrm{NA}(Z, Y)$ is dense in $\mathcal{L}(Z, Y)$ for any Banach space Z.

Gowers' counterexample

Theorem (Gowers (1990))

For any $1 < p < \infty$, the space ℓ_p does not satisfy property B. In particular, $\mathrm{NA}(d_*(w,1),\ell_p)$ *is not dense in* $\mathcal{L}(d_*(w,1),\ell_p)$.

Gowers' counterexample

Theorem (Dantas, Falcó, Jung, R-V (2023))

Let $w = (1/n)_{n=1}^{\infty} \in c_0$ *and* $1 < p < \infty$. *Then*

$$\mathcal{L}(d_*(w,1), \ell_p) \setminus \overline{\mathrm{NA}(d_*(w,1), \ell_p)}$$

is $\mathfrak{c}$*-spaceable in* $\mathcal{L}(d_*(w,1), \ell_p)$.

Gowers' counterexample

Theorem (Dantas, Falcó, Jung, R-V (2023))

Let $w = (1/n)_{n=1}^{\infty} \in c_0$ *and* $1 < p < \infty$*. Then*

$$\mathcal{L}(d_*(w,1),\ell_p) \setminus \overline{\mathrm{NA}(d_*(w,1),\ell_p)}$$

is $\mathfrak{c}$*-spaceable in* $\mathcal{L}(d_*(w,1),\ell_p)$*.*

Corollary (Dantas, Falcó, Jung, R-V (2023))

Let $w = (1/n)_{n=1}^{\infty} \in c_0$, $1 < p < \infty$ *and* $\alpha \geq \aleph_0$ *a cardinal number. Then,*

$$\mathcal{L}(d_*(w,1),\ell_p) \setminus \overline{\mathrm{NA}(d_*(w,1),\ell_p)}$$

is not (α,β)*-spaceable in* $\mathcal{L}(d_*(w,1),\ell_p)$ *regardless of the cardinal number* $\beta \geq \alpha$*.*

Bibliography

[1] R. M. Aron, L. Bernal González, D. M. Pellegrino, and J. B. Seoane Sepúlveda, *Lineability: the search for linearity in mathematics*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016.

[2] P. Bandyopadhyay and G. Godefroy, *Linear structures in the set of norm-attaining functionals on a Banach space*, J. Convex Anal. **13** (2006), no. 3-4, 489–497.

[3] L. Cheng and S. Luo, *Yet on linear structures of norm-attaining functionals on Asplund spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) **38** (2018), no. 1, 151–156.

[4] J. Falcó, S. Dantas, J. Mingu, and D. L. Rodríguez-Vidanes, *Linear structures in the set of non-norm-attaining operators on Banach spaces*, Preprint (2023).

[5] V. V. Fávaro, D. Pellegrino, A. Raposo Jr., and G. Ribeiro, *General criteria for a strong notion of lineability*, Proc. Amer. Math. Soc. **152** (2024), 941–954.

[6] P. Leonetti, T. Russo, and J. Somaglia, *Dense lineability and spaceability in certain subsets of* ℓ_∞, Bull. Lond. Math. Soc. **55** (2023), no. 5, 2283–2303.

[7] F. J. García-Pacheco and D. Puglisi, *Lineability of functionals and renormings*, Bull. Belg. Math. Soc. Simon Stevin **25** (2018), no. 1, 141–147.

[8] M. Rmoutil, *Norm-attaining functionals need not contain 2-dimensional subspaces*, J. Funct. Anal. **272** (2017), no. 3, 918–928.

Thank you for your attention!