Complemented copies of separable $\mathcal{C}(K)$ spaces in Banach spaces (based on a joint work in progress with Damian Sobota)

Jakub Rondoš

Faculty of Mathematics, University of Vienna

jakub.rondos@gmail.com

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Notation and the main problem

- Compact spaces are Hausdorff and infinite.
- If K is compact space, then $\mathcal{C}(K)$ stands for the Banach space of continuous real-valued functions on K endowed with the supremum norm.
- The symbol χ_A stands for the characteristic function of a set $A \subset K$, and ϵ_x stands for the Dirac measure at the point $x \in K$.

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Problem

Let E be a Banach space and K be a compact metric space (i.e., $\mathcal{C}(K)$ is separable). We are looking for a characterization of the presence of a complemented subspace of E which is isomorphic to $\mathcal{C}(K)$.

Cantor-Bendixon derivatives and scattered spaces

K compact space, α ordinal, then the *Cantor-Bendixon derivative* of K of order α is

$$K^{(0)} = K,$$
$$K^{(1)} = \{x \in K : x \text{ is an accumulation point of } K\},$$
$$K^{(\alpha)} = (K^{\beta})^{(1)}, \quad \alpha = \beta + 1,$$
$$K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}, \quad \alpha \text{ limit.}$$

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K is *scattered* if there exists an ordinal α such that $K^{(\alpha)} = \emptyset$. In such a case, the *height* of K is the smallest ordinal such that $K^{(\alpha)} = \emptyset$ (and it is a successor ordinal). If K is not scattered, then $ht(K) = \infty$ (we use the convention that $\alpha < \infty$ for each ordinal α).

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A compact metric space K is scattered iff it is countable.

Isomorphisms classes of separable $\mathcal{C}(K)$ spaces

- For a compact metric space K, $\mathcal{C}(K)$ is isomorphic either to $\mathcal{C}(2^\omega)$ (if K is uncountable) or to exactly one of the spaces $\mathcal{C}([1, \omega^{\omega^\alpha}])$ for some countable ordinal α (if K is countable).

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- 2^ω...the Cantor space (i.e., cardinal exponentiation)
- $[1, \omega^{\omega^\alpha}]$...the countable ordinal interval (i.e., ordinal exponentiation)
- c_0 represents the simplest isomorphism class of separable $\mathcal{C}(K)$ spaces: For a compact metric space K, $\mathcal{C}(K)$ is isomorphic to c_0 iff $K^{(n)} = \emptyset$ for some $n \in \mathbb{N}$.

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Fact

A Banach space E contains a complemented copy of c_0 iff there exist a sequence $(e_n)_{n\in\mathbb{N}}$ in E equivalent to the canonical basis of c_0 and a weak*-null sequence $(e_n^*)_{n\in\mathbb{N}}$ in E^* such that, for each $n, m \in \mathbb{N}$, $e_n^*(e_m) = \delta_{n,m}$.

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Our aim is to get a similar characterization for other $\mathcal{C}(K)$ spaces over metric compacta. For this, we need some technical preparation.

Trees

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- the empty sequence $\emptyset \in \Lambda$,
- Λ is closed with respect to taking initial segments, that is, whenever $t \in \Lambda$ and $s \prec t$, then $s \in \Lambda$,
- Λ does not have infinite branches,
- for each $s \in \Lambda$, either Λ is a leaf (i.e., $t \in \Lambda$ and $s \prec t$ implies $s = t$), or for each $n \in \mathbb{N}$, $s^\wedge n \in \Lambda$ (we identify n with the sequence $\langle n \rangle$),
- for each $s \in NL(\Lambda)$(i,e., s is not a leaf), the sequence of ordinals $(r(s^\wedge n))_{n\in\mathbb{N}}$ is either constant or strictly increasing, where the *rank* $r(s)$ of an element $s \in \Lambda$ is defined recursively as follows:
  If s is a leaf, then $r(s)$ is 0, otherwise $r(s) = \sup_{s\prec t}(r(t) + 1)$.
- The *rank* of $\emptyset$ is equal to $\alpha - 1$.

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By a *tree of rank* ∞ we mean the full tree $\omega^{<\omega}$.

Trees of continuous functions in Banach spaces

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- for each $s \in NL(\Lambda)$ and for each $e \in E$,

$$\lim_{n\to\infty} \sup_{s^\wedge n \prec t} |e_s^*(e) - e_t^*(e)| = 0.$$

Example

Proposition

For each countable ordinal α*, the space* $\mathcal{C}([1,\omega^\alpha])$ *admits a tree of continuous functions of rank* $\alpha+1$ *(recall that ht*$([1,\omega^\alpha]) = \alpha+1$*). The space* $\mathcal{C}(2^\omega)$ *admits a tree of continuous functions of rank* ∞*.*

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In $\mathcal{C}([1, \omega^2])$, the tree of continuous functions $(e_s, e_s^*)_{s \in \Lambda} \subseteq \mathcal{C}([1, \omega^2]) \times M([1, \omega^2])$ of rank 3 has the form

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- $e_\emptyset = \chi_{[1,\omega^2]}$, $e_\emptyset^* = \epsilon_{\omega^2}$
- $e_{\langle n\rangle} = \chi_{[\omega(n-1),\omega n]}$, $e_{\langle n\rangle}^* = \epsilon_{\omega n}$, $\quad n \in \mathbb{N}$
- $e_{\langle n,m\rangle} = \chi_{\{\omega(n-1)+m\}}$, $e_{\langle n,m\rangle}^* = \epsilon_{\omega(n-1)+m}$, $\quad n, m \in \mathbb{N}$

The main result

Theorem

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In the rest of the talk, we present an application of the result for the case when E is also a space of continuous functions.

An application to $\mathcal{C}(K)$ spaces

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- Even when the Banach space E is isometric to $\mathcal{C}(K)$ for some compact K, the question of complementability of spaces $\mathcal{C}(L)$ in it, where L is compact metric, is highly nontrivial:

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  Recall that a $\mathcal{C}(K)$ space contains a complemented copy of c_0 iff $\mathcal{C}(K)$ is not a Grothendieck Banach space (i.e., iff there exists a sequence of elements of $\mathcal{C}(K)^*$ that converges weak* but not weakly).

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- E.g., $\mathcal{C}(\beta\mathbb{N})$ does not contain a complemented copy of c_0.
- On the other hand, if K contains a convergent sequence then $\mathcal{C}(K)$ contains a complemented copy c_0.
- Notice that a convergent sequence A in K is a first-countable subspace (in the relative topology) such that $A^{(1)} \neq \emptyset$ (recall that a topological space is *first-countable* if each of its points has a countable neighborhood basis).

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Let K be a compact space, A be a first-countable subspace of K (in the relative topology), and α be a countable ordinal or ∞, such that $A^{(\alpha)} \neq \emptyset$. Then $\mathcal{C}(K)$ admits a tree of continuous functions of rank $\alpha + 1$. Consequently, for each compact metric space L such that $ht(L) \leq \alpha + 1$, $\mathcal{C}(K)$ contains a complemented subspace **isometric** *to $\mathcal{C}(L)$.*

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In particular, if a compact space K contains a non-scattered first-countable subspace, then $\mathcal{C}(K)$ contains a complemented isometric copy of $\mathcal{C}(L)$ for each compact metric space L.

More consequences-isometric version

Theorem

For a first-countable compact space K and a compact metric space L, the following assertions are equivalent:

(i) $\mathcal{C}(L)$ *is isometric to a complemented subspace of* $\mathcal{C}(K)$,

(ii) $\mathcal{C}(L)$ *is isometric to a subspace of* $\mathcal{C}(K)$,

(iii) $\mathcal{C}(L)$ *is isomorphically embedded into* $\mathcal{C}(K)$ *by an isomorphism with distortion less than* 3,

(iv) $ht(K) \geq ht(L)$, *and, if* L *is scattered, then* $\left|K^{(ht(L)-1)}\right| \geq \left|L^{(ht(L)-1)}\right|$.

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Proof.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, and (iii) implies (iv) by a theorem of Gordon, 1970.

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(iv) implies (i) follows from the previous result applied to $A = K$. □

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For a first-countable compact space K and a compact metric space L, the following assertions are equivalent:

(i) $\mathcal{C}(L)$ *is isomorphic to a complemented subspace of* $\mathcal{C}(K)$,

(ii) $\mathcal{C}(L)$ *is isomorphic to a subspace of* $\mathcal{C}(K)$,

(iii) $Sz(\mathcal{C}(K)) \geq Sz(\mathcal{C}(L))$ *(the Szlenk indices).*

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Proof.

(i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) is a property of the Szlenk index. (iii) $\Rightarrow$ (i) can be deduced from the previous result combined with the isomorphic classification of spaces of continuous functions over metric compacta. □

Acknowledgement

Thank you.