Special Ultrafilters, L-orthogonality, and octahedral norms

Luis Sáenz

Universidad Nacional Autónoma de México

July 10, 2024

Some definitions

Definition

Let $(X, \|\cdot\|)$ be a Banach space

1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in S_X$, $\epsilon > 0$ *there is* $y \in B_X$ *such that* $\|x_i + y\| > 2 - \epsilon$.

Some definitions

Definition

Let $(X, \|\cdot\|)$ *be a Banach space*

1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in S_X$, $\epsilon > 0$ *there is* $y \in B_X$ *such that* $\|x_i + y\| > 2 - \epsilon$.
2. $y^{**} \in X^{**}$ *is an* **L-orthogonal element** *iff for any* $x \in X$, $\|x + y^{**}\| = \|x\| + 1$.

Some definitions

Definition

Let $(X, \|\cdot\|)$ *be a Banach space*

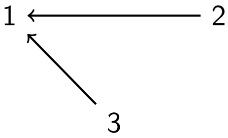
1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in S_X$, $\epsilon > 0$ *there is* $y \in B_X$ *such that* $\|x_i + y\| > 2 - \epsilon$.
2. $y^{**} \in X^{**}$ *is an* **L-orthogonal element** *iff for any* $x \in X$, $\|x + y^{**}\| = \|x\| + 1$.
3. $(y_n)_{n\in\mathbb{N}} \subseteq X^{**}$ *is an* **L-orthogonal sequence** *iff for any* $x \in X$, $\lim_{n\to\infty} \|x + y_n\| = \|x\| + 1$.

Some definitions

Definition

Let $(X, \|\cdot\|)$ *be a Banach space*

1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in S_X$, $\epsilon > 0$ *there is* $y \in B_X$ *such that* $\|x_i + y\| > 2 - \epsilon$.
2. $y^{**} \in X^{**}$ *is an* **L-orthogonal element** *iff for any* $x \in X$, $\|x + y^{**}\| = \|x\| + 1$.
3. $(y_n)_{n\in\mathbb{N}} \subseteq X^{**}$ *is an* **L-orthogonal sequence** *iff for any* $x \in X$, $\lim_{n\to\infty} \|x + y_n\| = \|x\| + 1$.

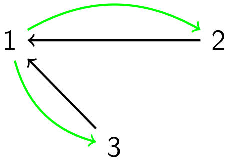


Some definitions

Definition

Let $(X, \|\cdot\|)$ *be a Banach space*

1. $\|\cdot\|$ *is* **octahedral** *if for any* $x_0, \ldots, x_{n-1} \in S_X$, $\epsilon > 0$ *there is* $y \in B_X$ *such that* $\|x_i + y\| > 2 - \epsilon$.
2. $y^{**} \in X^{**}$ *is an* **L-orthogonal element** *iff for any* $x \in X$, $\|x + y^{**}\| = \|x\| + 1$.
3. $(y_n)_{n\in\mathbb{N}} \subseteq X^{**}$ *is an* **L-orthogonal sequence** *iff for any* $x \in X$, $\lim_{n\to\infty} \|x + y_n\| = \|x\| + 1$.



Renorming results

Theorem (Godefroy, 1989 [2])

Let X be a Banach space, tfae,

- *X contains a isomorphic copy of ℓ_1.*
- *X admits an equivalent norm such that X^{**} has an L-orthogonal element.*

Renorming results

Theorem (Godefroy, 1989 [2])

Let X be a Banach space, tfae,

- *X contains a isomorphic copy of ℓ_1.*
- *X admits an equivalent norm such that X^{**} has an L-orthogonal element.*

Theorem (Kadets, Shepelska, Werner, 2011 [4])

Let X be a Banach space, tfae,

- *X contains a isomorphic copy of ℓ_1.*
- *X admits an equivalent norm such that X has an L-orthogonal sequence.*

Isometric results: Octahedrality and L-orthogonal elements

Theorem (López-Pérez, Rueda Zoca, 2021 [5])

Let X be a Banach space with $dens(X) \leq \omega_1$, if X has an octahedral norm, then there is an L-orthogonal element in X^{**}.

Isometric results: Octahedrality and L-orthogonal elements

Theorem (López-Pérez, Rueda Zoca, 2021 [5])

Let X be a Banach space with $dens(X) \leq \omega_1$, if X has an octahedral norm, then there is an L-orthogonal element in X^{**}.

Theorem (López-Pérez, Rueda Zoca, 2021)

There is a Banach space X with $dens(X) = 2^{\mathfrak{c}}$, and an octahedral norm such that X^{**} admits no L-orthogonal elements.

Isometric results: L-orthogonal elements and sequences

Proposition (Avilés, Martínez-Cervantez, Rueda Zoca, 2022 [1])

There is a Banach space X with no L-orthogonal sequence such that X^{**} admits a L-orthogonal element.

Isometric results: L-orthogonal elements and sequences

Proposition (Avilés, Martínez-Cervantez, Rueda Zoca, 2022 [1])

There is a Banach space X with no L-orthogonal sequence such that X^{**} admits a L-orthogonal element.

1. Let X be a Banach space with an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$, is there an L-orthogonal element $x \in X^{**}$?

Isometric results: L-orthogonal elements and sequences

Proposition (Avilés, Martínez-Cervantez, Rueda Zoca, 2022 [1])

There is a Banach space X with no L-orthogonal sequence such that X^{**} admits a L-orthogonal element.

1. Let X be a Banach space with an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$, is there an L-orthogonal element $x \in X^{**}$?
2. Let X be a Banach space with an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$, is there an L-orthogonal element $x^{**} \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$?

Isometric results: L-orthogonal elements and sequences

Proposition (Avilés, Martínez-Cervantez, Rueda Zoca, 2022 [1])

There is a Banach space X with no L-orthogonal sequence such that X^{**} admits a L-orthogonal element.

1. Let X be a Banach space with an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$, is there an L-orthogonal element $x \in X^{**}$?
2. Let X be a Banach space with an L-orthogonal sequence $(x_n)_{n\in\mathbb{N}}$, is there an L-orthogonal element $x^{**} \in \overline{\{x_n : n \in \mathbb{N}\}}^{w^*}$?

Avilés, et. al.:

The answer to (2) is independent of the usual Set Theory axioms (ZFC).

Main Result

Theorem (Hrušák - S, 2024 [3])

A free ultrafilter $\mathscr{U}$ on $\mathbb{N}$ is a Q-point if and only if for every Banach space X and every L-orthogonal sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ the $\mathscr{U}\text{-}\lim x_n \in X^{**}$ with respect to the weak* topology is an L-orthogonal element.

Ultrafilters

Definition (Filters)

A family $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ *is a* **filter over** $\mathbb{N}$ *if*

1. $\varnothing \notin \mathscr{F}$
2. *If* $F \in \mathscr{F}$, $F \subseteq E$ *then* $E \in \mathscr{F}$
3. *If* $F, F' \in \mathscr{F}$, *then* $F \cap F' \in \mathscr{F}$

- *A filter over* $\mathbb{N}$ *is* **free** *if* $\forall F \in \mathscr{F}$, $|F| = \aleph_0$.
- *A filter is an* **ultrafilter** *if it is a* $\subseteq$*-maximal filter.*

Ultrafilters

Definition (Filters)

A family $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N})$ *is a* **filter over** $\mathbb{N}$ *if*

1. $\varnothing \notin \mathscr{F}$
2. *If* $F \in \mathscr{F}$, $F \subseteq E$ *then* $E \in \mathscr{F}$
3. *If* $F, F' \in \mathscr{F}$, *then* $F \cap F' \in \mathscr{F}$

- *A filter over* $\mathbb{N}$ *is* **free** *if* $\forall F \in \mathscr{F}$, $|F| = \aleph_0$.
- *A filter is an* **ultrafilter** *if it is a* $\subseteq$*-maximal filter.*

We will consider $\mathscr{P}(\mathbb{N})$ equipped with the natural topology inherited from the product topology of $2^{\mathbb{N}}$ via characteristic functions. So we can talk of topological properties as: closed, Borel, analytic, etc.

Ideals

Definition (Ideals)

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is an* **ideal over** $\mathbb{N}$ *if*

1. $\mathbb{N} \notin \mathcal{I}$
2. *If* $I \in \mathcal{I}$, $J \subseteq I$ *then* $J \in \mathcal{I}$
3. *If* $I, I' \in \mathcal{F}$, *then* $I \cup I' \in \mathcal{I}$
4. *An ideal over* $\mathbb{N}$ *is* **free** *if* $\forall I \subseteq \mathbb{N}$ *finite,* $I \in \mathcal{I}$.

- *An ideal* $\mathcal{I}$ *is* **countably hitting** *if for any countable family of infinite subsets of* $\mathbb{N}$, $\{A_n : n \in \mathbb{N}\}$, *there is* $I \in \mathcal{I}$ *such that for any* $n \in \mathbb{N}$, $A_n \cap I$ *is infinite.*

Special Ultrafilters

Definition ($\mathscr{F}$-limit)

Given a topological space X *a filter* $\mathscr{F}$ *over* $\mathbb{N}$*, and a sequence* $(x_n)_{n\in\mathbb{N}}$*, the* $\mathscr{F}$*-limit with respect to the sequence is* $x \in X$ *(*$x = \mathscr{F}\text{-}\lim x_n$*) iff for every* V *neighborhood of* x

$$\{n \in \mathbb{N} : x_n \in V\} \in \mathscr{F}$$

Special Ultrafilters

Definition

An ultrafilter $\mathscr{U}$ over $\mathbb{N}$ is a Q-point if for any partition $\{A_n : n \in \omega\}$ of $\mathbb{N}$ into finite pieces there is $F \in \mathscr{U}$ such that for any $n \in \mathbb{N}$, $|A_n \cap F| = 1$.

Corolary (Hrušák-S, 2024)

Let $\mathcal{U}$ be an ultrafilter, the following are equivalent:

1. *$\mathscr{U}$ is a Q-point.*
2. *For every F_σ ideal $\mathcal{I}$ that is* **countably hitting**, *$\mathscr{U} \cap \mathcal{I} \neq \varnothing$.*
3. *For every analytic ideal $\mathcal{I}$ that is* **countably hitting**, *$\mathscr{U} \cap \mathcal{I} \neq \varnothing$.*

$\mathcal{U} - \lim$ of Q points are L-orthogonal

The important object

Let X be a Banach space, $(x_n)_{n\in\mathbb{N}}$ a sequence in B_X, $(\epsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to zero, $Z \subseteq X$ a subspace and $(F_n)_{n\in\mathbb{N}}$ an increasing sequence of finite dimensional subspaces of X such that $Z = \overline{\bigcup_{n\in\mathbb{N}} F_n}$.

$\mathcal{L}_{(F_n)_{n\in\mathbb{N}}}$ is the family of all $B \subseteq \mathbb{N}$ with the following property:

For every $n \in \mathbb{N}$, and any $A \subseteq B$ such that

$$n \leq |\{m \in B : m \leq \min A\},$$

if $w \in \overline{conv}\{x_n : n \in A\}$ and $y \in F_n$, then

$$(1 - \epsilon_n)(1 + \|y\|) \leq \|y + w\|\}$$

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is the ideal generated by this family.

$\mathcal{U}-$ lim of Q points are L-orthogonal

Proof.

$\mathcal{U} - \lim$ of Q points are L-orthogonal

Proof. Start with a L-orthogonal sequence $(x_n)_{n\in\omega}$. Let $\mathscr{U}$ be a Q-point.

$\mathcal{U} - \lim$ of Q points are L-orthogonal

Proof. Start with a L-orthogonal sequence $(x_n)_{n\in\omega}$. Let $\mathscr{U}$ be a Q-point.
Due to $B_{X^{**}}$ being w^*-compact, the $\mathcal{U} - \lim x_n$ exists, call it u.

$\mathcal{U} - \lim$ of Q points are L-orthogonal

Proof. Start with a L-orthogonal sequence $(x_n)_{n\in\omega}$. Let $\mathscr{U}$ be a Q-point.
Due to $B_{X^{**}}$ being w^*-compact, the $\mathcal{U} - \lim x_n$ exists, call it u.
We will show u is an L-orthogonal element.

$\mathcal{U} - \lim$ of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

$\mathcal{U}$ − lim of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u + y\| = 1 + \|y\|$.

$\mathcal{U}-\lim$ of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u+y\| = 1 + \|y\|$.

Step 2. Prove that

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is a countably hitting F_σ ideal.

$\mathcal{U}-\lim$ of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u+y\| = 1 + \|y\|$.

Step 2. Prove that

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is a countably hitting F_σ ideal.

Finally, consider an arbitrary $y \in X$, and $Z = \langle y \rangle$.

$\mathcal{U}-\lim$ of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u+y\| = 1 + \|y\|$.

Step 2. Prove that

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is a countably hitting F_σ ideal.

Finally, consider an arbitrary $y \in X$, and $Z = \langle y \rangle$.
By $\mathscr{U}$ being a Q-point and step 2: $\mathscr{U} \cap \mathcal{I}_{\langle x \rangle} \neq \varnothing$.

$\mathcal{U}-\lim$ of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u+y\| = 1 + \|y\|$.

Step 2. Prove that

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is a countably hitting F_σ ideal.

Finally, consider an arbitrary $y \in X$, and $Z = \langle y \rangle$.
By $\mathscr{U}$ being a Q-point and step 2: $\mathscr{U} \cap \mathcal{I}_{\langle x\rangle} \neq \varnothing$.
By u being the $\mathscr{U}$-lim, for any $B \in \mathscr{U}$,
$u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$.

$\mathcal{U}-$ lim of Q points are L-orthogonal

Consider all the elements necessary to define $\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ for any arbitrary separable Z subspace of X.

Step 1. Prove that

If $B \in \mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ and $u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$ then for any $y \in Z$, $\|u + y\| = 1 + \|y\|$.

Step 2. Prove that

$\mathcal{I}_{(F_n)_{n\in\mathbb{N}}}$ is a countably hitting F_σ ideal.

Finally, consider an arbitrary $y \in X$, and $Z = \langle y \rangle$.
By $\mathscr{U}$ being a Q-point and step 2: $\mathscr{U} \cap \mathcal{I}_{\langle x\rangle} \neq \varnothing$.
By u being the $\mathscr{U}$- lim, for any $B \in \mathscr{U}$,
$u \in \bigcap_{n\in\mathbb{N}} \overline{conv}^{w^*}\{x_m : m \in B, m \geq n\}$.
Therefore, $\|u + y\| = 1 + \|y\|$.

References I

[1] Antonio Avilés, Gonzalo Martínez-Cervantes, and Abraham Rueda Zoca. "L-Orthogonal Elements and L-Orthogonal Sequences". In: *International Mathematics Research Notices* 2023.11 (May 2022), pp. 9128–9154.

[2] Gilles Godefroy. "Metric characterization of first Baire class linear forms and octahedral norms". In: *Studia Mathematica* 95.1 (1989), pp. 1–15.

[3] Michael Hrušák and Luis Sáenz. "Some applications of Q-points and Lebesgue filters to Banach Spaces". In: *Extracta Mathematicae* (2024).

[4] Vladimir Kadets, Varvara Shepelska, and Dirk Werner. "Thickness of the unit sphere, ℓ_1-types, and the almost Daugavet property". In: *Houston journal of mathematics* 37 (Jan. 2011), pp. 867–878.

References II

Special Ultrafilters, L-orthogonality, and octahedral norms

Luis Sáenz

[5] Ginés López-Pérez and Abraham Rueda Zoca. "L-orthogonality, octahedrality and Daugavet property in Banach spaces". In: *Advances in Mathematics* 383 (2021), p. 107719. ISSN: 0001-8708.