Functionals on Lipschitz spaces and Choquet representation theory

New perspectives in Banach spaces and Banach lattices, CIEM Castro Urdiales

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Lipschitz and Lipschitz-free Banach spaces

Definition 1

1. Let (M, d) be a complete metric space with base point 0. Define the **Lipschitz space** $\mathrm{Lip}_0(M)$ to be the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ that vanish at 0, with norm

$$\|f\| := \mathrm{Lip}(f) = \sup\left\{ \frac{f(x) - f(y)}{d(x,y)} \; : \; x, y \in M,\, x \neq y \right\}.$$

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2. Define $\widetilde{M} = \{(x, y) \in M^2 \;:\; x \neq y\}$ and the set $\mathrm{Mol} = \left\{m_{xy} \;:\; (x, y) \in \widetilde{M}\right\} \subseteq S_{\mathrm{Lip}_0(M)^*}$ of **(elementary) molecules** m_{xy}, where

$$\langle f, m_{xy}\rangle = \frac{f(x) - f(y)}{d(x, y)}, \quad f \in \mathrm{Lip}_0(M).$$

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3. Define the **(Lipschitz-) free** Banach space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}(\mathrm{Mol}) \subseteq \mathrm{Lip}_0(M)^*.$$

Basic facts about free spaces

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Example 3

There is a linear isometric embedding $T : L_1 \to \mathcal{F}(\mathbb{R})$, given by

$$\langle g, Tf \rangle = \int_{-\infty}^{\infty} f(t) g'(t) \, \mathrm{d}t, \quad g \in \mathrm{Lip}_0(\mathbb{R}).$$

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The **De Leeuw transform** is the isometric embedding $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$, defined by

$$(\Phi f)(x, y) = \langle f, m_{xy} \rangle , \quad (x, y) \in \widetilde{M},$$

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Definition 5

Let $\psi \in \mathrm{Lip}_0(M)^*$. We define the set

$$\mathcal{M}_{\mathrm{op}}(\psi) = \left\{\mu \in \mathcal{M}(\beta\widetilde{M}) \;:\; \Phi^*\mu = \psi,\, \mu \geqslant 0 \text{ and } \|\psi\| = \|\Phi^*\mu\| = \|\mu\|\right\}$$

of **optimal (De Leeuw) representations** of ψ.

Examples of optimal representations

Example 6

Let $(x, y) \in \widetilde{M}$. Then $\delta_{(x,y)} \in \mathcal{M}_{\mathrm{op}}(m_{xy})$ as $\Phi^*\delta_{(x,y)} = m_{xy}$ and $\left\|\delta_{(x,y)}\right\| = 1 = \|m_{xy}\|$.

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Example 7

Let $M := [0, 1]$ have base point 0. Given $n \geqslant 0$, define positive $\mu_n \in \mathcal{M}(\beta\widetilde{M})$ by

$$\mu_n = \frac{1}{2^n}\sum_{i=1}^{2^n} \delta_{\left(\frac{i}{2^n},\frac{i-1}{2^n}\right)}.$$

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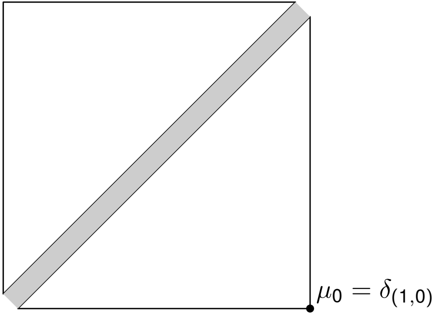
Then $\mu_n \in \mathcal{M}_{\mathrm{op}}(m_{10})$: $\|\mu_n\| = 1 = \|m_{10}\|$ and

$$\langle f, \Phi^*\mu_n\rangle = \langle \Phi f, \mu_n\rangle = \frac{1}{2^n}\sum_{i=1}^{2^n} \frac{f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right)}{2^{-n}} = f(1) - f(0) = \langle f, m_{10}\rangle, \quad f \in \mathrm{Lip}_0(M),$$

giving $\Phi^*\mu_n = m_{10}$.

Examples of optimal representations

Below is a depiction of $\beta\widetilde{M}$, with the shaded area representing the remainder $\beta\widetilde{M} \setminus \widetilde{M}$.



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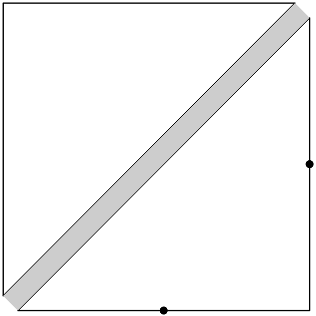

$$\mu_1 = \frac{1}{2}\left(\delta_{(1,\frac{1}{2})} + \delta_{(\frac{1}{2},0)}\right)$$

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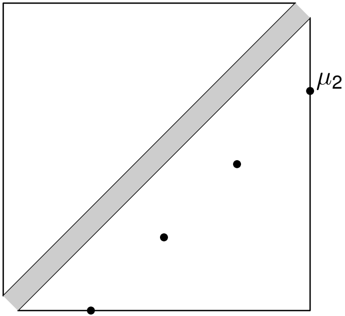

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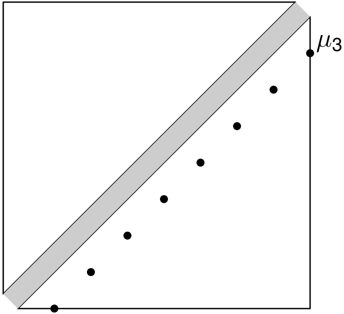

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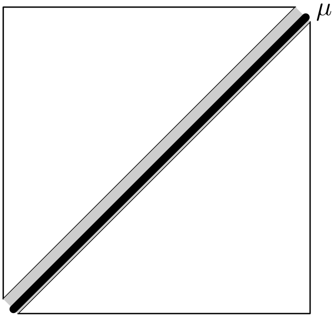

Any w^*-cluster point μ of (μ_n) also belongs to $\mathcal{M}_{\mathrm{op}}(m_{10})$; any such measure is supported entirely on $\beta\widetilde{M} \setminus \widetilde{M}$.

Weaver's extreme point conjecture

Recall that $\mathrm{Mol} = \left\{ m_{xy} \ : \ (x,y) \in \widetilde{M} \right\}$ is the set of elementary molecules of $\mathcal{F}(M)$.

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Conjecture 9 (Weaver mid-90s)

Every extreme point of $B_{\mathcal{F}(M)}$ is an elementary molecule: $\mathrm{ext}\, B_{\mathcal{F}(M)} \subseteq \mathrm{Mol}$.

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Recall $\mathcal{F}(\mathbb{R}) \equiv L_1$, so sometimes $\mathrm{ext}\, B_{\mathcal{F}(M)} = \varnothing$.

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Example 12

Let $C \subseteq [0,1] = M$ be a fat Cantor set, recall the isometry $T : L_1 \to \mathcal{F}(\mathbb{R})$ and set $m = T\mathbf{1}_C \in \mathcal{F}(\mathbb{R})$.

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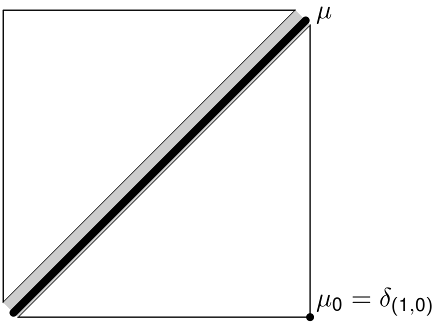
Then $\mu(\widetilde{M}) = 0$ whenever $\mu \in \mathcal{M}_{\mathrm{op}}(m)$.

Some optimal representations are better than others



Recall $\mu_0, \mu \in \mathcal{M}_{\mathrm{op}}(m_{10})$. The measures μ_0 and μ are supported on $\widetilde{M}$ and $\beta\widetilde{M} \setminus \widetilde{M}$, respectively, so $\mu_0(\widetilde{M}) = 1$ and $\mu(\widetilde{M}) = 0$.

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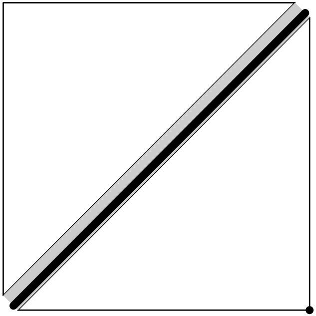

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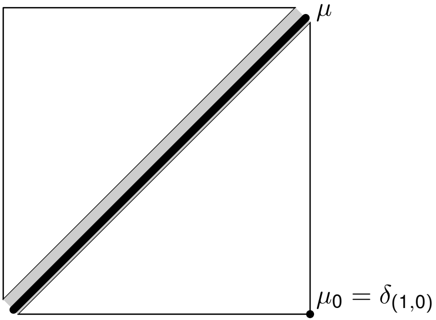

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Can we make this intuition precise and, if so, what can this tell us about the structure of $\mathrm{Lip}_0(M)^*$ and $\mathcal{F}(M)$?

A characterisation of $\Phi\,\mathrm{Lip}_0(M) \subseteq C(\beta\widetilde{M})$

Proposition 13

$g \in \Phi\,\mathrm{Lip}_0(M) \subseteq C(\beta\widetilde{M})$ if and only if

$$d(x,y)g(x,y) = d(x,p)g(x,p) + d(p,y)g(p,y) \quad \text{whenever } x,p,y \in M \text{ are distinct.} \qquad (*)$$

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Sketch proof:

Let $f \in \mathrm{Lip}_0(M)$. Given distinct $x,p,y \in M$:

$$\begin{aligned} d(x,y)(\Phi f)(x,y) &= f(x) - f(y) \\ &= f(x) - f(p) + f(p) - f(y) = d(x,p)(\Phi f)(x,p) + d(p,y)(\Phi f)(p,y). \end{aligned}$$

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Conversely, given $g \in C(\beta\widetilde{M})$ satisfying $(*)$, define $f : M \to \mathbb{R}$ by

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Then $f \in \mathrm{Lip}_0(M)$ and $g = \Phi f$. □

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Define G to be the set of all $g \in C(\beta\widetilde{M})$ satisfying

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3. $\mathbf{1} \in G$ by the triangle inequality.

A quasi-ordering on $\mathcal{M}(\beta\widetilde{M})^+$

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Define $\preccurlyeq$ on $\mathcal{M}(\beta\widetilde{M})^+$ by $\mu \preccurlyeq \nu$ if and only if $\langle g, \mu\rangle \leqslant \langle g, \nu\rangle$ for all $g \in G$.

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3. $\preccurlyeq$ is ‘anti-symmetric enough’ (as $\overline{G - G}$ is ‘big enough’).
4. If $\nu \in \mathcal{M}_{\mathrm{op}}(\psi)$ and $\mu \preccurlyeq \nu$, then $\mu \in \mathcal{M}_{\mathrm{op}}(\psi)$.

Minimal measures and comparison with Choquet theory

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We say that $\mu \in \mathcal{M}(\beta\widetilde{M})^+$ is **minimal** if $\lambda \preccurlyeq \mu$ implies $\mu \preccurlyeq \lambda$.

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2. In Choquet theory the focus is on **maximal** measures, which are 'concentrated' on the **Choquet boundary** (analogous to the set of extreme points of a compact convex set).
3. But this boundary can be a subset of the remainder $\beta\widetilde{M} \setminus \widetilde{M}$, which we want to avoid.

'Coordinates' of $\beta\widetilde{M}$ and marginals

Let $M^{\mathcal{U}}$ denote the uniform compactification of M, and define its 'Lipschitz realcompactification' by

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$$\mathfrak{p}_1(x,y) = x, \quad \mathfrak{p}_2(x,y) = y, \quad (x,y) \in \widetilde{M},$$

and then extending continuously to $\beta\widetilde{M}$.

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1. Given $\mu \in \mathcal{M}(\beta\widetilde{M})^+$, define its **marginals** as the push-forwards $(\mathfrak{p}_1)_\sharp\mu, (\mathfrak{p}_2)_\sharp\mu \in \mathcal{M}(M^{\mathcal{U}})$:

$$(\mathfrak{p}_1)_\sharp(E) = \mu(\mathfrak{p}_1^{-1}(E)), \quad (\mathfrak{p}_2)_\sharp(E) = \mu(\mathfrak{p}_2^{-1}(E)), \quad E \subseteq M^{\mathcal{U}} \text{ Borel.}$$

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2. The marginals of μ are **mutually singular** if $(\mathfrak{p}_1)_\sharp\mu \perp (\mathfrak{p}_2)_\sharp\mu$.

Mutually singular marginals and convex integrals of molecules

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If $m \in \mathcal{F}(M)$ and $\mu \in \mathcal{M}_{\mathrm{op}}(m)$, then μ is concentrated on R.

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If $m \in \mathcal{F}(M)$, and $\mu \in \mathcal{M}_{\mathrm{op}}(m)$ is concentrated on $\mathrm{R} \setminus d^{-1}(0)$, is minimal, and moreover

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then μ is concentrated on $\widetilde{M}$. Consequently m is a convex integral of molecules.

Back to extreme points

Corollary 25

If M is **uniformly discrete** ($\inf d\restriction_{\widetilde{M}} = \inf_{x \neq y} d(x,y) > 0$), then $\mathrm{ext}\, B_{\mathcal{F}(M)} \subseteq \mathrm{Mol}$.

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Sketch proof: Let $\mu \in \mathcal{M}(\beta\widetilde{M})$. As $\min d = \inf d\restriction_{\widetilde{M}} > 0$, $\displaystyle\int_{\beta\widetilde{M}} \frac{1}{d(\zeta)}\, d\mu(\zeta) \leqslant \frac{\|\mu\|}{\min d} < \infty$.

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Example 26

Let $M = \bigcup_{n=1}^{\infty} K_n$, such that each K_n is finite, $\operatorname{diam}(K_n) \to 0$ and $\inf_{m \neq n} d(K_m, K_n) > 0$. Then $\operatorname{ext} B_{\mathcal{F}(M)} \subseteq \mathrm{Mol}$.