Complemented subspaces of Banach spaces $C(K \times L)$

Damian Sobota

Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Joint work with Grzegorz Plebanek and Jakub Rondoš.

Simple Question

K, L always stand for infinite compact spaces

Simple Question

K, L always stand for infinite compact spaces

Simple Question

What are non-trivial complemented subspaces of $C(K \times L)$?

Simple Question

K, L always stand for infinite compact spaces

Simple Question

What are non-trivial complemented subspaces of $C(K \times L)$?

Trivial complemented subspaces

$C(K)$ and $C(L)$ are complemented in $C(K \times L)$:

Simple Question

K, L always stand for infinite compact spaces

Simple Question

What are non-trivial complemented subspaces of $C(K \times L)$?

Trivial complemented subspaces

$C(K)$ and $C(L)$ are complemented in $C(K \times L)$:

$T\colon C(K) \to C(K \times L)$

$C(K) \ni f \mapsto T(f)(x, y) = f(x)$ for every $(x, y) \in K \times L$

T is an isometry onto a complemented subspace of $C(K \times L)$

Simple Question

K, L always stand for infinite compact spaces

Simple Question

What are non-trivial complemented subspaces of $C(K \times L)$?

Trivial complemented subspaces

$C(K)$ and $C(L)$ are complemented in $C(K \times L)$:

$T \colon C(K) \to C(K \times L)$

$C(K) \ni f \mapsto T(f)(x, y) = f(x)$ for every $(x, y) \in K \times L$

T is an isometry onto a complemented subspace of $C(K \times L)$

Fact

If a Banach space E is complemented in $C(K)$ or $C(L)$, then E is complemented in $C(K \times L)$.

Cembranos–Freniche Theorem

Theorem (Cembranos, Freniche '84)

c_0 is complemented in $C(K \times L)$.

Cembranos–Freniche Theorem

Theorem (Cembranos, Freniche '84)

c_0 is complemented in $C(K \times L)$.

Observation

$C(K)$ contains an isometric copy of c_0.

Cembranos–Freniche Theorem

Theorem (Cembranos, Freniche '84)

c_0 is complemented in $C(K \times L)$.

Observation

$C(K)$ contains an isometric copy of c_0.

Theorem (Phillips '40)

c_0 is not complemented in $\ell_\infty \cong C(\beta\mathbb{N})$.

Cembranos–Freniche Theorem

Theorem (Cembranos, Freniche '84)

c_0 is complemented in $C(K \times L)$.

Observation

$C(K)$ contains an isometric copy of c_0.

Theorem (Phillips '40)

c_0 is not complemented in $\ell_\infty \cong C(\beta\mathbb{N})$.

Theorem (Räbiger, Cembranos, Schachermayer,...)

c_0 is not complemented in $C(K) \Leftrightarrow C(K)$ is *Grothendieck* (i.e. weak* convergent sequences in $C(K)^*$ are weakly convergent).

Cembranos–Freniche Theorem

Theorem (Cembranos, Freniche '84)

c_0 is complemented in $C(K \times L)$.

Observation

$C(K)$ contains an isometric copy of c_0.

Theorem (Phillips '40)

c_0 is not complemented in $\ell_\infty \cong C(\beta\mathbb{N})$.

Theorem (Räbiger, Cembranos, Schachermayer,...)

c_0 is not complemented in $C(K) \Leftrightarrow C(K)$ is *Grothendieck* (i.e. weak* convergent sequences in $C(K)^*$ are weakly convergent).

Examples

If $K \in \{$metric, scattered, Rosenthal, Eberlein, Corson,...$\}$, then c_0 is complemented in $C(K)$.

Pointwise Topology Interlude

X, Y stand for infinite Tychonoff spaces

$C_p(X)$ is $C(X)$ with the pointwise topology inherited from $\mathbb{R}^X$

$(c_0)_p$ is c_0 with the pointwise topology inherited from $\mathbb{R}^{\mathbb{N}}$

Pointwise Topology Interlude

X, Y stand for infinite Tychonoff spaces

$C_p(X)$ is $C(X)$ with the pointwise topology inherited from $\mathbb{R}^X$

$(c_0)_p$ is c_0 with the pointwise topology inherited from $\mathbb{R}^{\mathbb{N}}$

Theorem (Banakh–Kąkol–Śliwa '19)

$(c_0)_p$ is complemented in $C_p(X)$ if and only if there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely supported signed measures on X such that $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$ and $\int_X f \, d\mu_n \to 0$ for every $f \in C(X)$.

Pointwise Topology Interlude

X, Y stand for infinite Tychonoff spaces

$C_p(X)$ is $C(X)$ with the pointwise topology inherited from $\mathbb{R}^X$

$(c_0)_p$ is c_0 with the pointwise topology inherited from $\mathbb{R}^{\mathbb{N}}$

Theorem (Banakh–Kąkol–Śliwa '19)

$(c_0)_p$ is complemented in $C_p(X)$ if and only if there is a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely supported signed measures on X such that $\|\mu_n\| = 1$ for every $n \in \mathbb{N}$ and $\int_X f \, d\mu_n \to 0$ for every $f \in C(X)$.

By the Closed Graph Theorem:

If $(c_0)_p$ is complemented in $C_p(K)$, then c_0 is complemented in $C(K)$.

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N} \times \beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N} \times \beta\mathbb{N})$, and so in any $C_p(K \times L)$.

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N} \times \beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N} \times \beta\mathbb{N})$, and so in any $C_p(K \times L)$. ($\Rightarrow$ Cembranos–Freniche theorem)

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N} \times \beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N} \times \beta\mathbb{N})$, and so in any $C_p(K \times L)$. ($\Rightarrow$ Cembranos–Freniche theorem)

X is *pseudocompact* if every $f \in C(X)$ is bounded

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N} \times \beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N} \times \beta\mathbb{N})$, and so in any $C_p(K \times L)$. ($\Rightarrow$ Cembranos–Freniche theorem)

X is *pseudocompact* if every $f \in C(X)$ is bounded

Theorem (Kąkol–Marciszewski–S.–Zdomskyy '22)

(1) If $X \times Y$ is pseudocompact, then $(c_0)_p$ is complemented in $C_p(X \times Y)$.

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N} \times \beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N} \times \beta\mathbb{N})$, and so in any $C_p(K \times L)$. ($\Rightarrow$ Cembranos–Freniche theorem)

X is *pseudocompact* if every $f \in C(X)$ is bounded

Theorem (Kąkol–Marciszewski–S.–Zdomskyy '22)

(1) If $X \times Y$ is pseudocompact, then $(c_0)_p$ is complemented in $C_p(X \times Y)$.

(2) Consistently, there is a pseudocompact space $X \subseteq \beta\mathbb{N}$ such that $(c_0)_p$ is not complemented in $C_p(X \times X)$.

Pointwise Topology Interlude, cont.

Theorem (Kąkol–S.–Zdomskyy '21)

There is a sequence $(\mu_n)_{n\in\mathbb{N}}$ as in the B–K–Ś theorem on $\beta\mathbb{N}\times\beta\mathbb{N}$.

In particular, $(c_0)_p$ is complemented in $C_p(\beta\mathbb{N}\times\beta\mathbb{N})$, and so in any $C_p(K\times L)$. ($\Rightarrow$ Cembranos–Freniche theorem)

X is *pseudocompact* if every $f\in C(X)$ is bounded

Theorem (Kąkol–Marciszewski–S.–Zdomskyy '22)

(1) If $X\times Y$ is pseudocompact, then $(c_0)_p$ is complemented in $C_p(X\times Y)$.

(2) Consistently, there is a pseudocompact space $X\subseteq\beta\mathbb{N}$ such that $(c_0)_p$ is not complemented in $C_p(X\times X)$.

(2) holds under CH, or MA, or $\mathfrak{d}=\mathfrak{c}\leqslant\mathfrak{u}^+$...

More Difficult Question

Question

What other non-trivial Banach spaces E are also complemented in $C(K \times L)$?

More Difficult Question

Question

What other non-trivial Banach spaces E are also complemented in $C(K \times L)$?

Particular Question (Alspach–Galego '11)

What are non-trivial complemented subspaces of $C(\beta\mathbb{N} \times \beta\mathbb{N})$?

More Difficult Question

Question

What other non-trivial Banach spaces E are also complemented in $C(K \times L)$?

Particular Question (Alspach–Galego '11)

What are non-trivial complemented subspaces of $C(\beta\mathbb{N} \times \beta\mathbb{N})$?

Theorem (Candido '23)

$c_0(C(\beta\mathbb{N})) := (\bigoplus_{n\in\mathbb{N}} C(\beta\mathbb{N}))_{c_0}$ is complemented in $C(\beta\mathbb{N} \times \beta\mathbb{N})$.

Main Result

Main Theorem (Plebanek–Rondoš–S. '24)

If K and L continuously map onto a compact topological group G, then $C(G)$ is complemented in $C(K \times L)$.

Main Result

Main Theorem (Plebanek–Rondoš–S. '24)

If K and L continuously map onto a compact topological group G, then $C(G)$ is complemented in $C(K \times L)$.

Theorem

If K and L continuously map onto $[0,1]^\kappa$, then $C([0,1]^\kappa)$ is complemented in $C(K \times L)$.

Main Result

Main Theorem (Plebanek–Rondoš–S. '24)

If K and L continuously map onto a compact topological group G, then $C(G)$ is complemented in $C(K \times L)$.

Theorem

If K and L continuously map onto $[0,1]^\kappa$, then $C([0,1]^\kappa)$ is complemented in $C(K \times L)$.

K is *scattered* if each subset $X \subseteq K$ has an isolated point

Main Result

Main Theorem (Plebanek–Rondoš–S. '24)

If K and L continuously map onto a compact topological group G, then $C(G)$ is complemented in $C(K \times L)$.

Theorem

If K and L continuously map onto $[0,1]^\kappa$, then $C([0,1]^\kappa)$ is complemented in $C(K \times L)$.

K is *scattered* if each subset $X \subseteq K$ has an isolated point

Theorem (Pełczyński–Semadeni '59)

K is non-scattered if and only if K maps onto $[0,1]$.

Main Result

Main Theorem (Plebanek–Rondoš–S. '24)

If K and L continuously map onto a compact topological group G, then $C(G)$ is complemented in $C(K \times L)$.

Theorem

If K and L continuously map onto $[0,1]^\kappa$, then $C([0,1]^\kappa)$ is complemented in $C(K \times L)$.

K is *scattered* if each subset $X \subseteq K$ has an isolated point

Theorem (Pełczyński–Semadeni '59)

K is non-scattered if and only if K maps onto $[0,1]$.

Corollary

If K and L are non-scattered, then $C([0,1])$ is complemented in $C(K \times L)$.

$C(\beta\mathbb{N} \times \beta\mathbb{N})$

Fact

If K is a separable compact space, then $\beta\mathbb{N}$ continuously maps onto K.

$C(\beta\mathbb{N} \times \beta\mathbb{N})$

Fact

If K is a separable compact space, then $\beta\mathbb{N}$ continuously maps onto K.

Theorem (Hewitt–Marczewski–Pondiczery)

$[0,1]^\kappa$ is separable for any $1 \leqslant \kappa \leqslant \mathfrak{c}$.

$C(\beta\mathbb{N} \times \beta\mathbb{N})$

Fact

If K is a separable compact space, then $\beta\mathbb{N}$ continuously maps onto K.

Theorem (Hewitt–Marczewski–Pondiczery)

$[0,1]^\kappa$ is separable for any $1 \leqslant \kappa \leqslant \mathfrak{c}$.

Corollary

For any $1 \leqslant \kappa \leqslant \mathfrak{c}$, $C([0,1]^\kappa)$ is complemented in $C(\beta\mathbb{N} \times \beta\mathbb{N})$.

$C(\beta\mathbb{N} \times \beta\mathbb{N})$

Fact

If K is a separable compact space, then $\beta\mathbb{N}$ continuously maps onto K.

Theorem (Hewitt–Marczewski–Pondiczery)

$[0,1]^\kappa$ is separable for any $1 \leqslant \kappa \leqslant \mathfrak{c}$.

Corollary

For any $1 \leqslant \kappa \leqslant \mathfrak{c}$, $C([0,1]^\kappa)$ is complemented in $C(\beta\mathbb{N} \times \beta\mathbb{N})$.

Theorem (Miljutin 66', Pełczyński '68)

For any metric K and $\kappa \geqslant \aleph_0$, $C(K^\kappa)$ is isomorphic to $C([0,1]^\kappa)$.

$C(\beta\mathbb{N} \times \beta\mathbb{N})$

Fact

If K is a separable compact space, then $\beta\mathbb{N}$ continuously maps onto K.

Theorem (Hewitt–Marczewski–Pondiczery)

$[0,1]^\kappa$ is separable for any $1 \leqslant \kappa \leqslant \mathfrak{c}$.

Corollary

For any $1 \leqslant \kappa \leqslant \mathfrak{c}$, $C([0,1]^\kappa)$ is complemented in $C(\beta\mathbb{N} \times \beta\mathbb{N})$.

Theorem (Miljutin 66', Pełczyński '68)

For any metric K and $\kappa \geqslant \aleph_0$, $C(K^\kappa)$ is isomorphic to $C([0,1]^\kappa)$.

Corollary

For any metric K and $1 \leqslant \kappa \leqslant \mathfrak{c}$, $C(K^\kappa)$ is complemented in $C(\beta\mathbb{N} \times \beta\mathbb{N})$.

The end

Thank you for the attention!