Most Iterations of Projections Converge

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$$C_1, C_2 \subseteq H \quad \textit{closed, convex subsets of Hilbert space}$$

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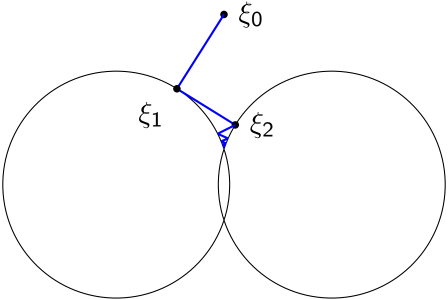
P_1 and P_2 nearest point projections onto C_1 and C_2.



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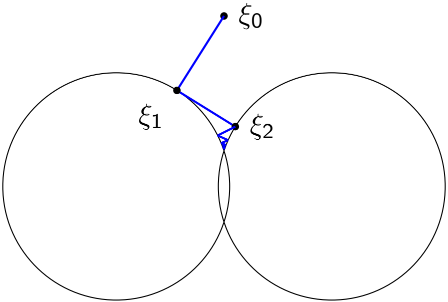
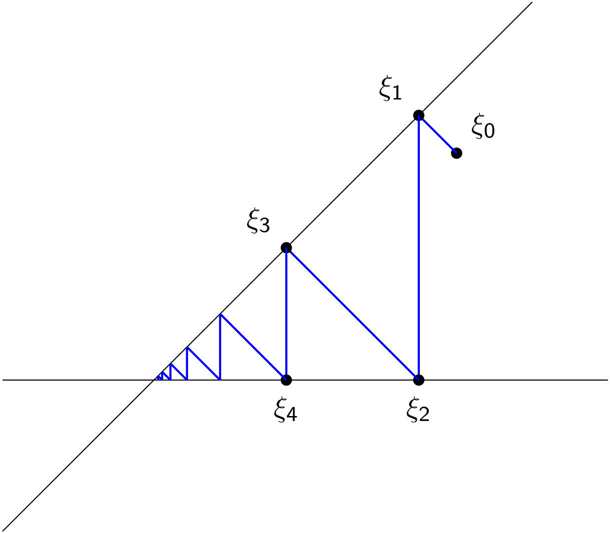
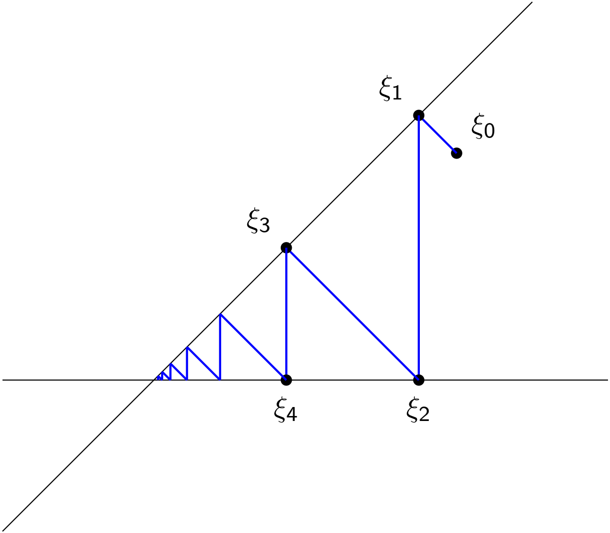


Hope that $(\xi_n)_{n\in\mathbb{N}}$ converges to some $p \in C_1 \cap C_2$.
Does it work?


ξ_1
ξ_0
ξ_3
ξ_4
ξ_2



Theorem (von Neumann, '49)

C_1, C_2 linear subspaces $\implies$ alternating projections work.
In fact: $\lim_{n\to\infty} \xi_n = P_{C_1 \cap C_2}(\xi_0)$

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Only periodic?

Definition (quasi-periodic)

A sequence $x \in \{1, \ldots, N\}^{\mathbb{N}}$ is **quasi-periodic** iff
$\exists m \in \mathbb{N}$ (the quasi period)
$\forall k \in \mathbb{N}$

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Idea: Don't let occurrences spread out too much.

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More than periodic?

All reasonable projection orders?

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Theorem (Kopecká, Müller, Paszkiewicz, '14, '17)

H infinite-dimensional
$\exists$ special choice of C_1, C_2, C_3 linear subspaces such that
$\forall 0 \neq \xi_0 \in H$ $\exists$ projection order x that
leads to a non-convergent projection series ξ_n.

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Definition (Measure on K)

Equip $I = \{1, \dots, N\}$ with Bernoulli measure

$$\mathbb{P}_I(\{1\}) = \cdots = \mathbb{P}_I(\{N\}) = \frac{1}{N}$$

and $K = I^{\mathbb{N}}$ with the infinite product measure $\mathbb{P}$ of $\mathbb{P}_I$.

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Theorem (Melo, da Cruz Neto, de Brito, '22)

$\mathbb{P}$-almost all orders $x \in K$ lead to $(\xi_n)_{n \in \mathbb{N}}$ being strongly convergent (under some constraints).

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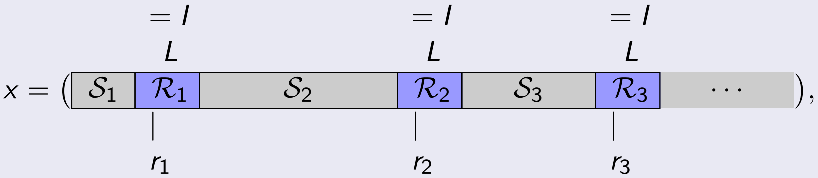

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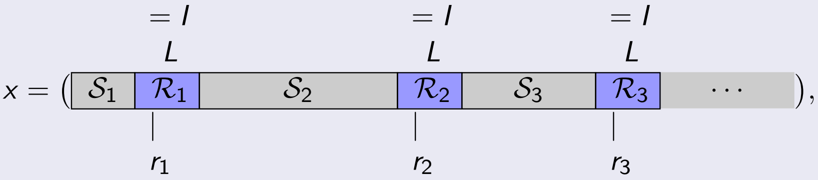

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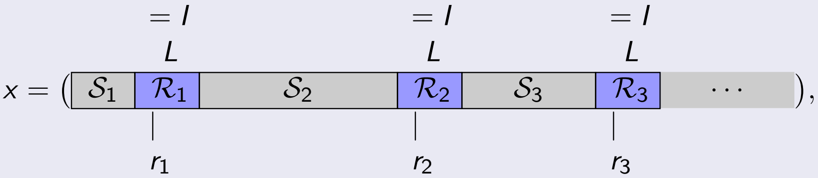

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$$\sum_{k \in \mathbb{N}} \frac{1}{r_k f(r_k)} = \infty.$$

Theorem (Melo, da Cruz Neto, de Brito, '22)

$$\left.\begin{array}{ll}\text{(i)} & x \text{ quasi-normal} \\ \text{(ii)} & (\xi_n)_{n\in\mathbb{N}} \text{ has accumulation point}\end{array}\right\} \textcircled{\wedge} \implies (\xi_n)_{n\in\mathbb{N}} \text{ converges}$$

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Guaranteed if:

- in Hilbert space
- one C_j compact, j in x infinitely often
- Hadamard manifold

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Definition (Metric on K)

On I choose discrete metric d_0.
On K choose

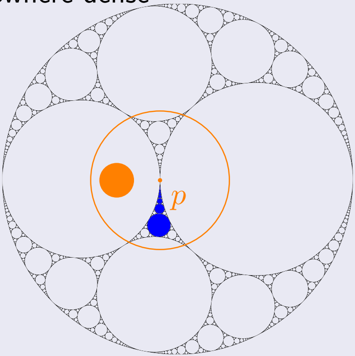
$$\begin{aligned} d(x, y) &:= \max\{2^{-j} d_0(x_j, y_j) \colon j \in \mathbb{N}\} \\ &= 2^{-(\text{first index where } x_j \neq y_j)}. \end{aligned}$$

Note that

$$B(x, 2^{-j}) = \{(x_1, \ldots, x_j, ?, ?, ?, ?, \ldots)\}.$$

Definition ((ϕ-)porosity)

Metric version of nowhere dense



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ϕ-**porous**	holes scale to given function
σ-(ϕ-)**porous**	countable union of ($\phi-$)porous sets.
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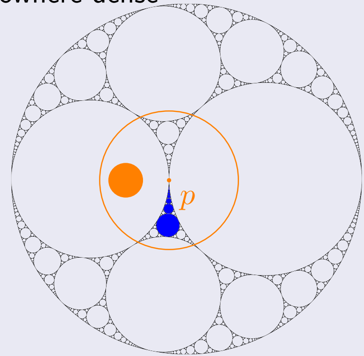

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Complement is large, **co-($\cdots$)-porous** $\implies$ dense G_δ.

How large is the set of sequences $x \in K$ for which $(\xi_n)_{n\in\mathbb{N}}$ is strongly convergent?

(in a metric sense)

Periodic sequences?

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Proposition (T., '23)

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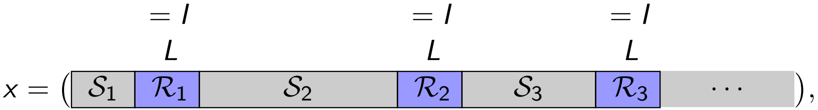

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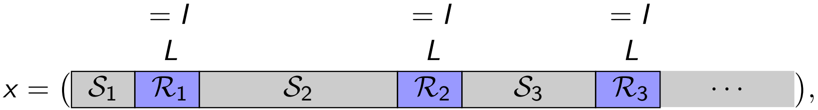

$$\mathcal{N}_0 := \left\{ x \in \mathcal{N} : \sum_{j=1}^{k} |\mathcal{S}_j| \leq ck \right\}$$

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Theorem (T., '23)

$$\mathcal{N}_0 \subseteq (K, \mathcal{T}) \text{ nowhere dense}$$

Quasi-normals, actually?

Theorem (T., '24)

$\mathcal{N} \subseteq (K, d)$ contains a co-σ-ϕ-porous subset

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$$\mathcal{N}$$

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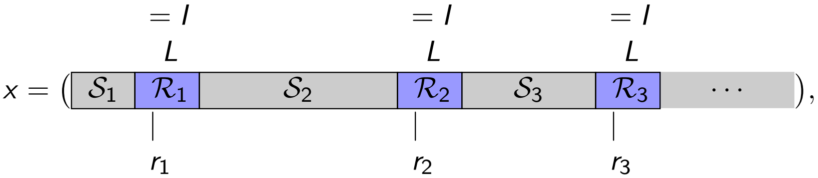

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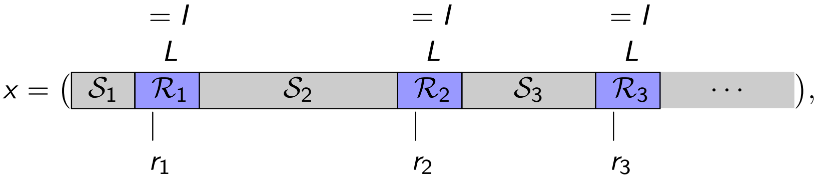

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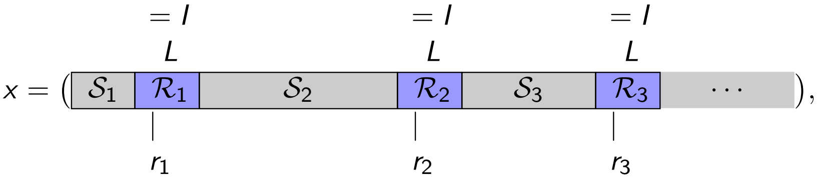

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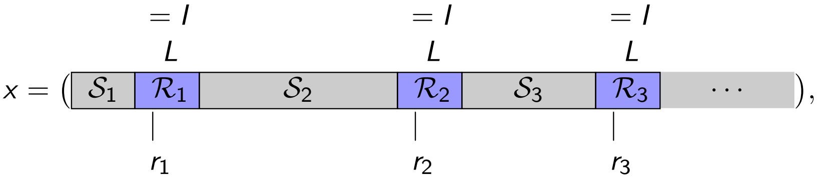
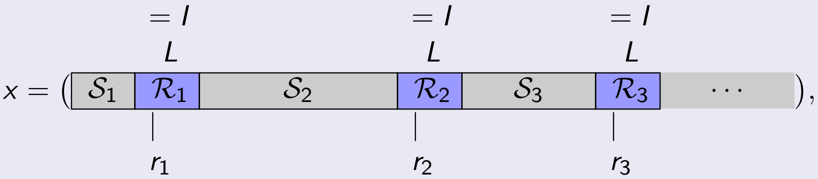


s.t.

$$\sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty.$$

Definition (Greedy L-partition)

$(r_k)_{k\in\mathbb{N}}$ greedy L-partition: Choose blocks $\mathcal{R}_k$ as far left as possible.



This maximizes

$$\sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty.$$

Definition

x quasi-normal $\iff$

$$\exists L \geq N \colon \bigcirc\!\!\!\!\wedge \begin{cases} \sum_{k\in\mathbb{N}} \frac{1}{r_k} = \infty & \text{(a)} \\ \text{greedy } L\text{-partition } (r_k)_{k\in\mathbb{N}} \text{ exists} & \text{(b)} \end{cases}$$

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Show that complement $K \setminus \mathcal{N}$ is small, σ-ϕ-porous.

$$x \notin \mathcal{N} \iff \forall L \geq N \colon \neg(a) \vee \neg(b)$$

Definition

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$$\exists L \geq N \colon \bigcirc\!\!\!\!\wedge \begin{cases} \sum_{k \in \mathbb{N}} \frac{1}{r_k} = \infty & \text{(a)} \\ \text{greedy } L\text{-partition } (r_k)_{k \in \mathbb{N}} \text{ exists} & \text{(b)} \end{cases}$$

Show that complement $K \setminus \mathcal{N}$ is small, σ-ϕ-porous.

$$x \notin \mathcal{N} \iff \forall L \geq N \colon \neg(a) \vee \neg(b)$$

$$x \in K \setminus \mathcal{N} \iff x \in \bigcap_{L \geq N} \underbrace{(A_L \cup B_L)}_{\text{small}}$$

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$$B(y, 2^{-s}) = (\underbrace{x_1, x_2, x_3, x_4, x_5, x_6, \overbrace{1, 1, 1, 1, 1, 1, 1}^{m}}_{s}, 1, 1, \ldots) \cap Q_m = \emptyset$$

□