Daugavet and Δ-points
Joint work with T. A. Abrahamsen, R. Aliaga, V. Lima, A. Martiny, Y. Perreau, and A. Prochazka.

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Preliminaries

We consider only real Banach spaces and use common notation.
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A **slice** of the unit ball B_X is a set

$$S(x^*, \alpha) = \{y \in B_X : x^*(y) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Daugavet property and diametral local diameter two property

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A Banach space X has the **Daugavet property**, if $\sup_{y \in S} \|x - y\| = 2$ for every $x \in S_X$ and for every slice S of B_X.

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For example $C[0,1]$ has the Daugavet property (Daugavet, 1963).

Daugavet points and Δ-points

Definition

Let X be a Banach space, and let $x \in S_X$. We say that x is

1. a *Daugavet point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X;
2. a *Δ-point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice S of B_X that contains the element x;

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3. a *super Daugavet point* if $\sup_{y \in V} \|x - y\| = 2$ for every non-empty relatively weakly open subset V of B_X;
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5. a *$\mathfrak{D}$-point* if $\sup_{y \in S} \|x - y\| = 2$ for every slice $S = S(x^*, \varepsilon)$ of B_X with $x^*(x) = 1$ and $\varepsilon > 0$.

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   In particular,
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   - Does there exist a renorming of ℓ_2 that contains a Δ-point?
2. Can every infinite dimensional Banach space be renormed to have a Daugavet point?

Duality and Δ-points

Theorem (Abrahamsen, Aliaga, Lima, Martiny, Perreau, Prochazka, V. 2024)

Let X be a Banach space. If X contains a $\mathfrak{D}$-point, or if X^* contains a weak Δ-point, then X^* contains a weak* super Δ-point.*

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Corollary (Abrahamsen, Aliaga, Lima, Martiny, Perreau, Prochazka, V. 2024)

If X is a reflexive Banach space with k-unconditional basis for $k < 2$, then X and X^* contain no $\mathfrak{D}$-points.

Renorming ℓ_2 with a Δ-point.

Following (Dilworth, Kutzarova, Randrianarivony, Revalski, Zhivkov, 2016) we introduce an equivalent norm on ℓ_2 by defining the unit ball by

$$B(\ell_2, \|\cdot\|) := \overline{\mathrm{conv}}(B_{\ell_2} \cup \{\pm(e_1 + e_n)_{n\geqslant 2}\}).$$

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$$|||x||| := \max\{\|x\|, \sup_{n\geqslant 2} |x_1 - 2x_n|\}.$$

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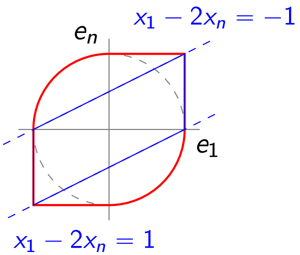

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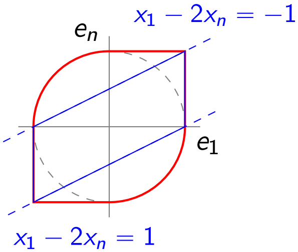

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Then $|||e_1||| = |||e_1 + e_n||| = 1$ and $|||e_1 - (e_1 + e_n)||| = |||e_n||| = 2$ for $n \geqslant 2$. Additionally, $e_1 + e_n \to e_1$ weakly. Thus e_1 is a super Δ-point.

Renorming any Banach space with a Δ-point.

Theorem (Abrahamsen, Aliaga, Lima, Martiny, Perreau, Prochazka, V. 2024)

Let X be a Banach space with weakly null basis. Then X can be renormed with a super Δ-point.

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Let X be an infinite dimensional Banach space with unconditional weakly null Schauder basis. Then X can be renormed with a super Daugavet point.

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Open question

- Can every infinite dimensional Banach space be renormed to have a Daugavet point?